Stam's Conjecture and Threshold Phenomena in Collision Resistance

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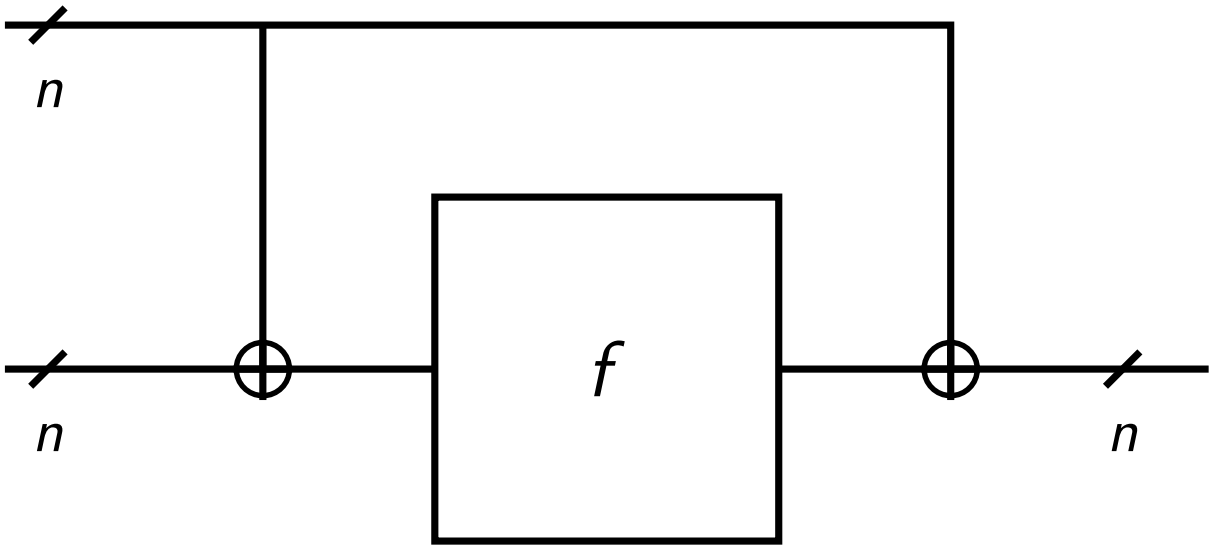
- An information-theoretic adversary is given oracle access to the primitive, which is modeled as "ideal" or "perfectly random".
- The only obstacle to the adversary's success is the randomness of the query responses.

The limits of IPM security



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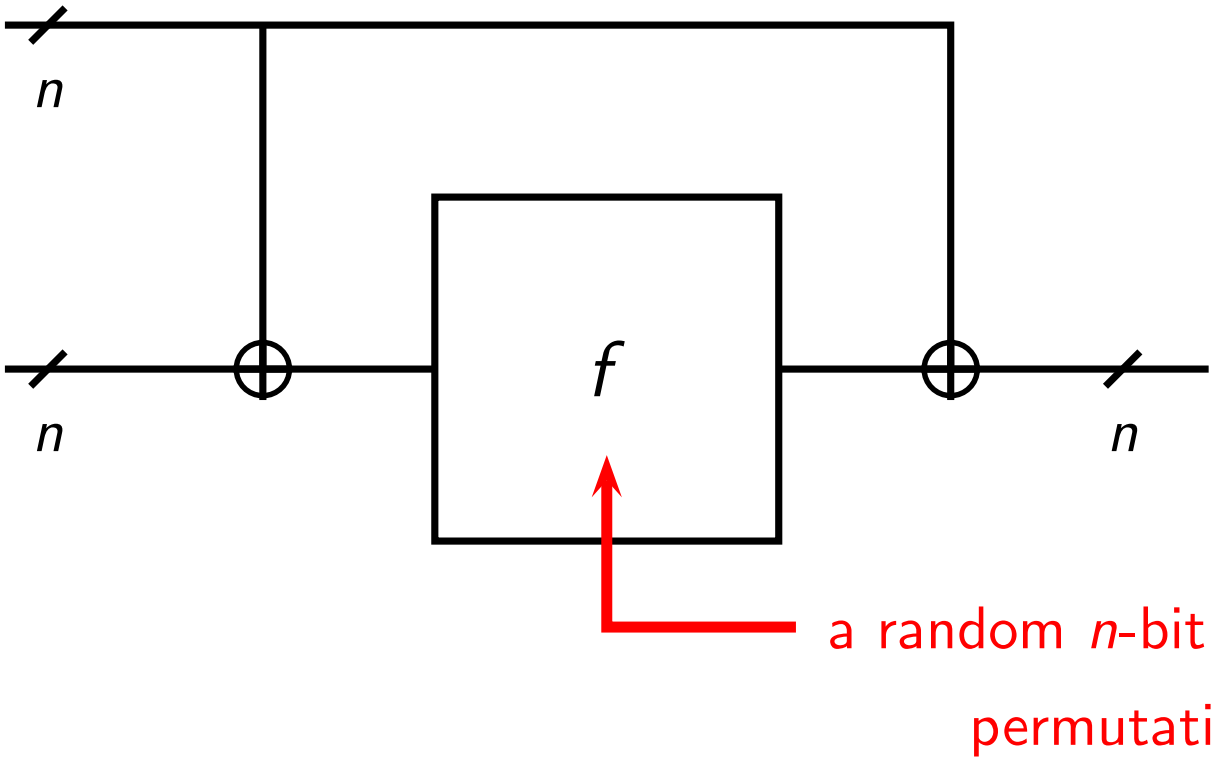

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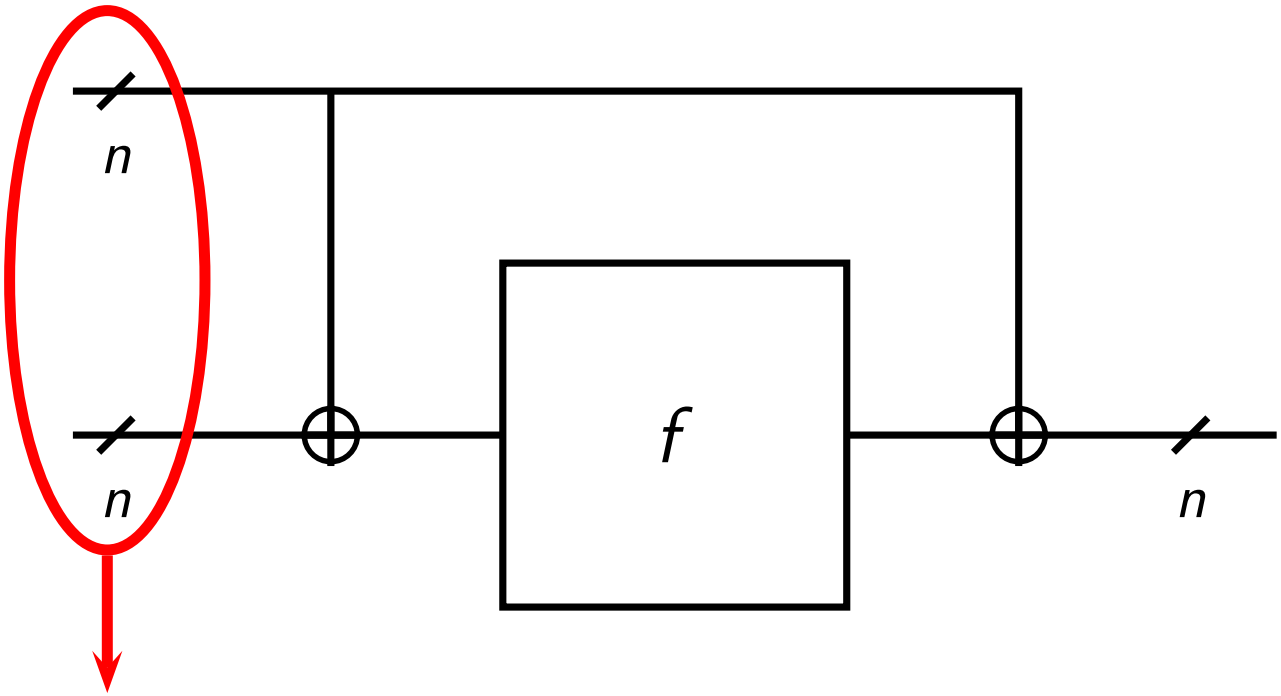

Domain of the compression function is $\{0,1\}^{2n}$

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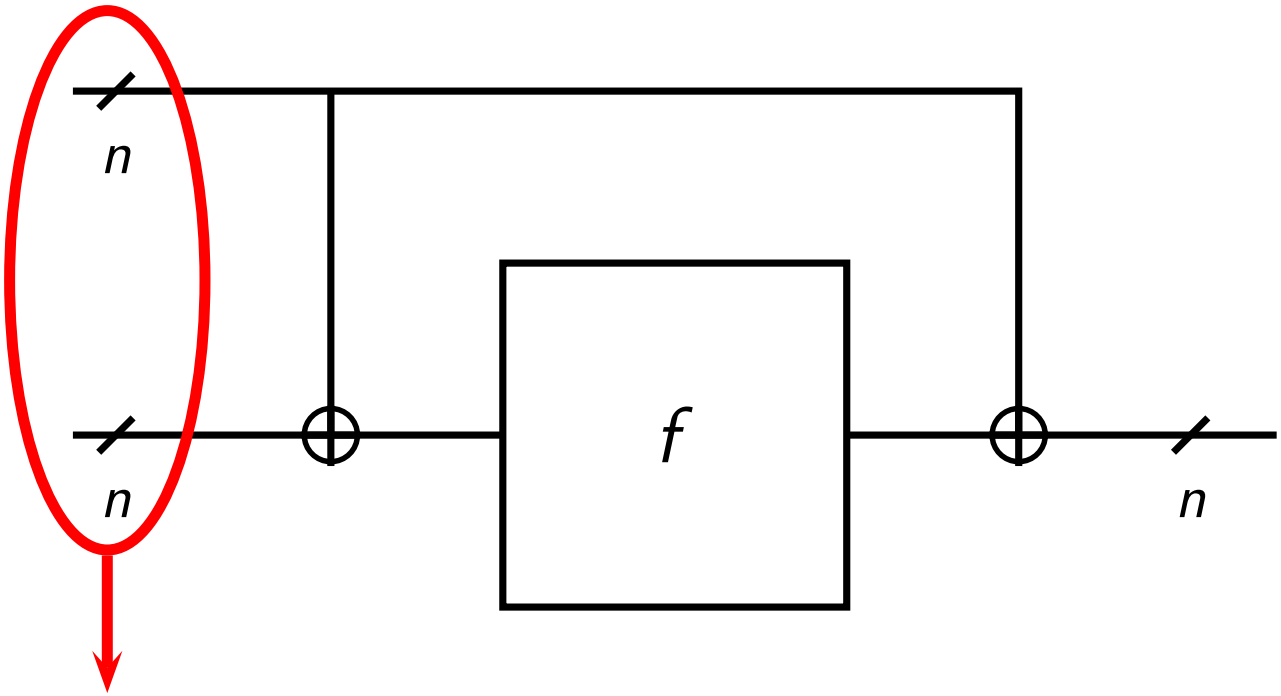

Domain of the compression function is $\{0,1\}^{2n}$

...the size of the domain is 2^{2n}

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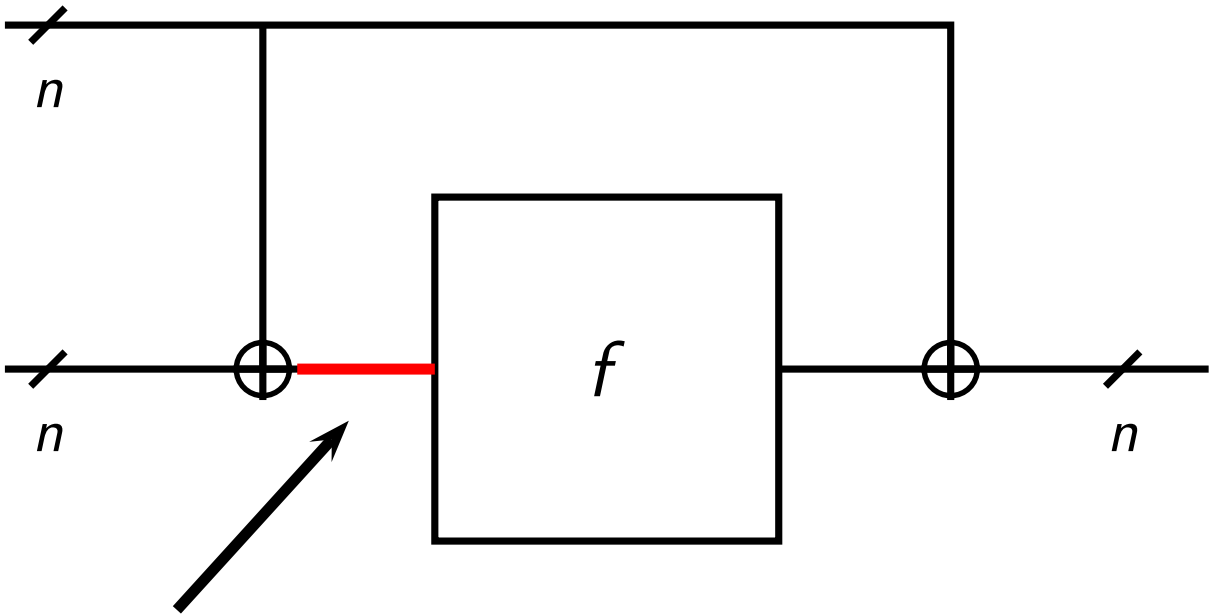

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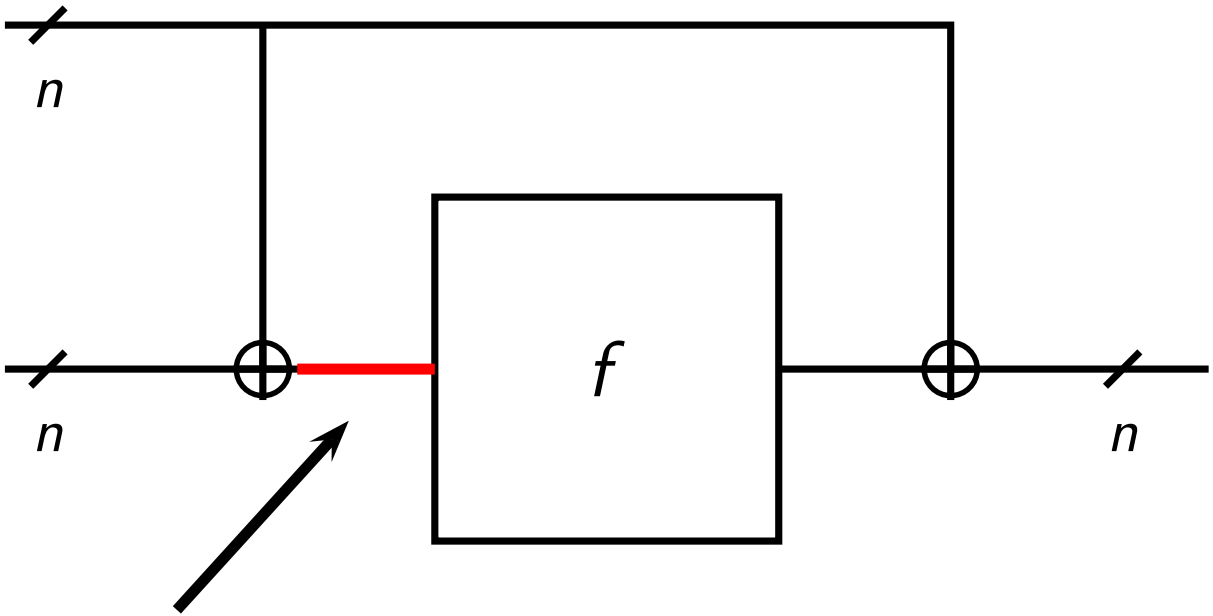

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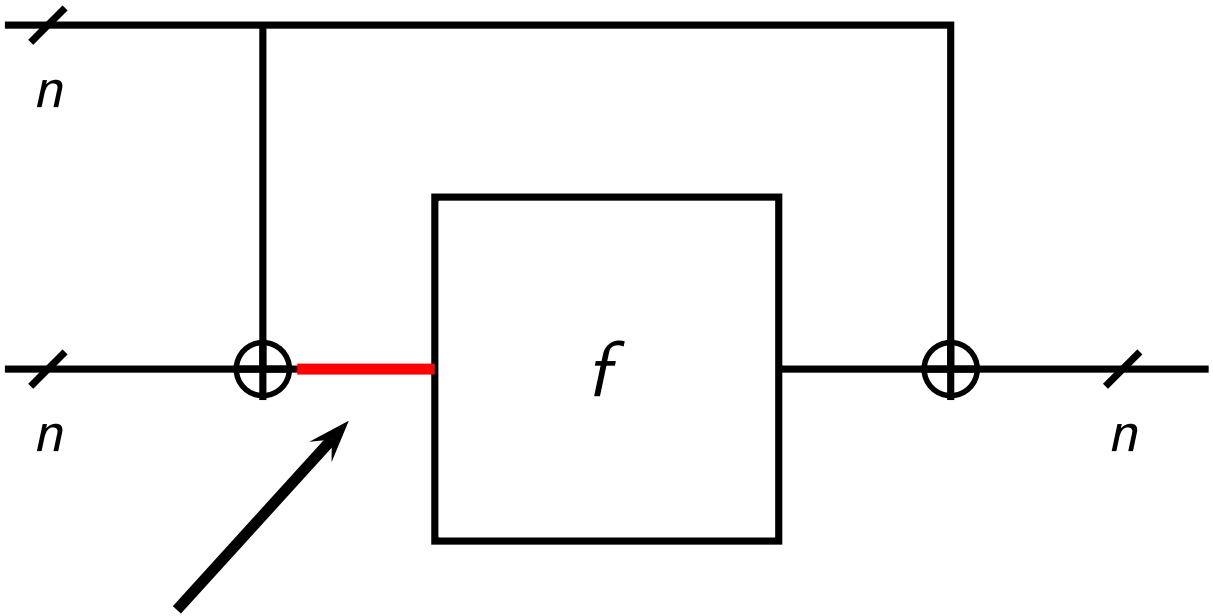

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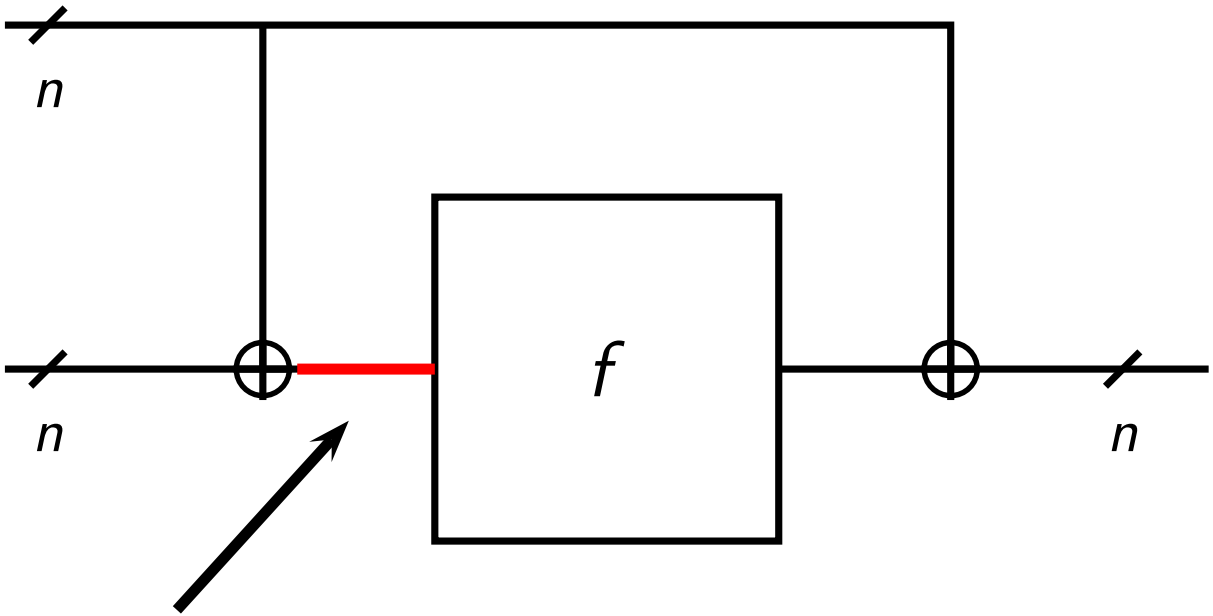

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...so the adversary breaks collision security in two queries.

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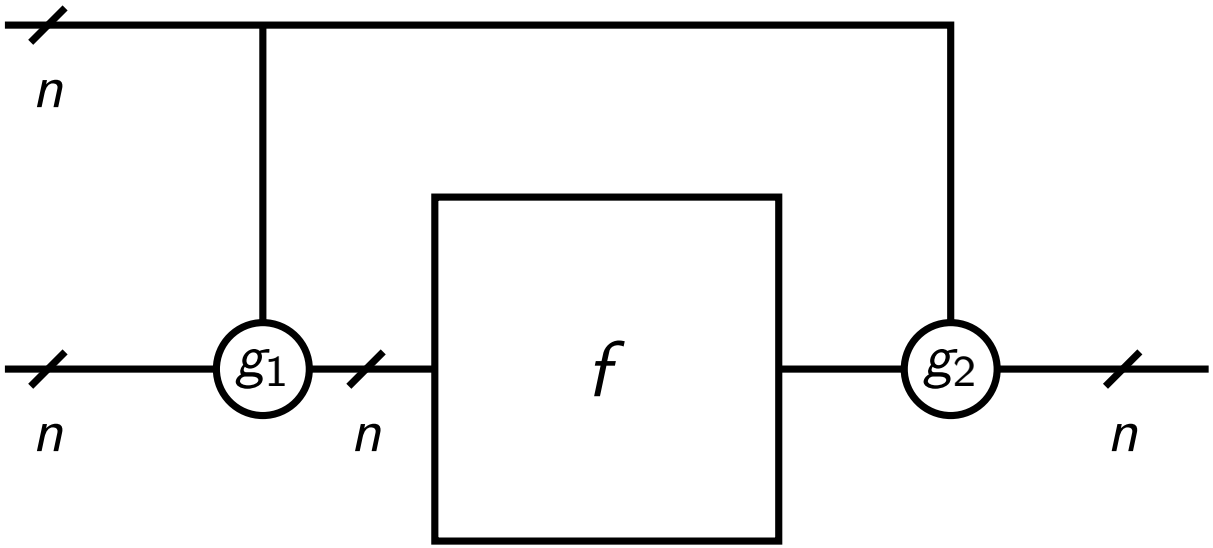

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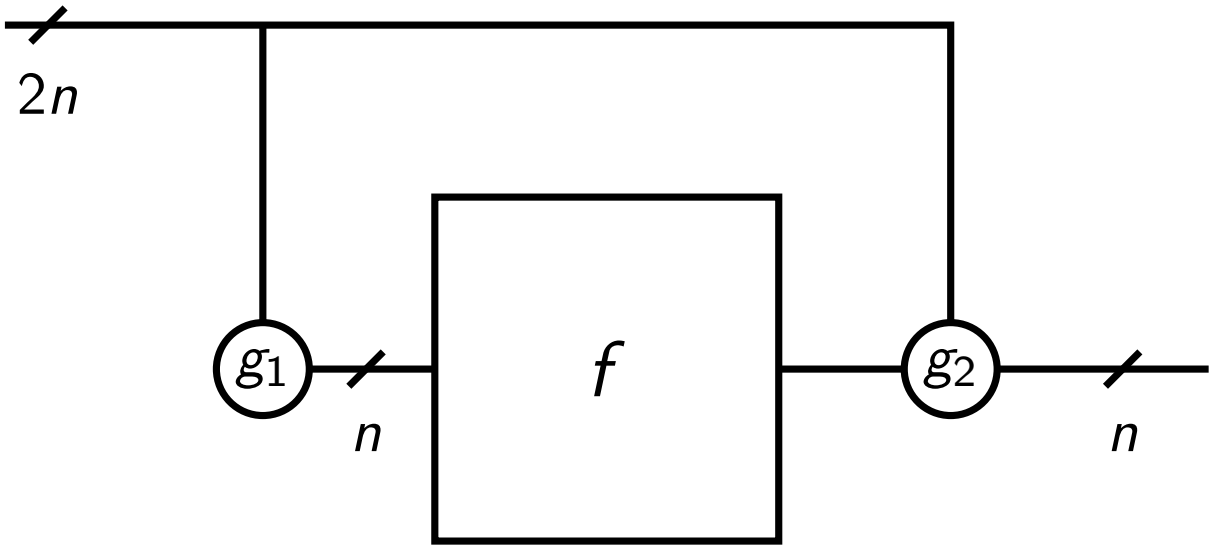

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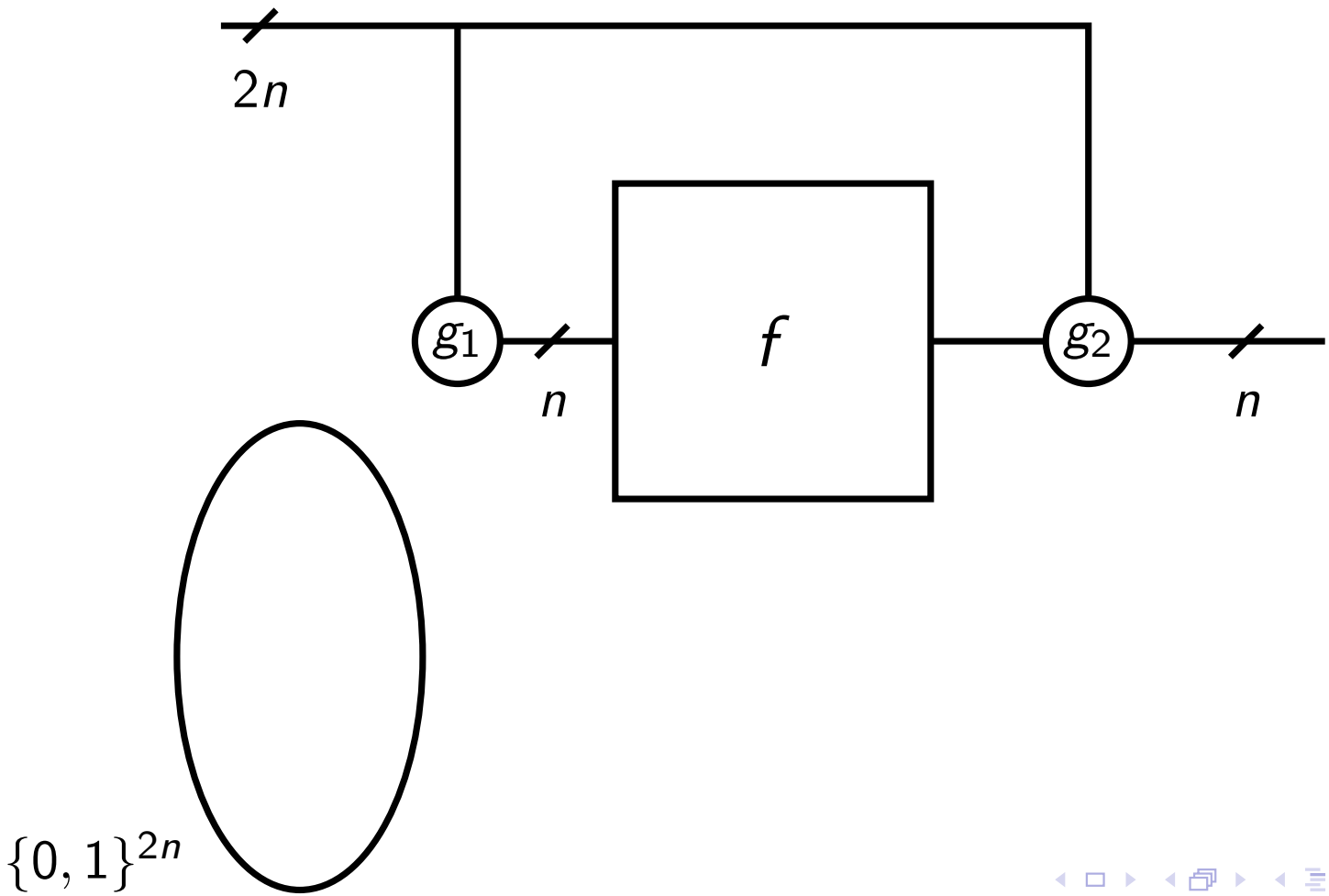

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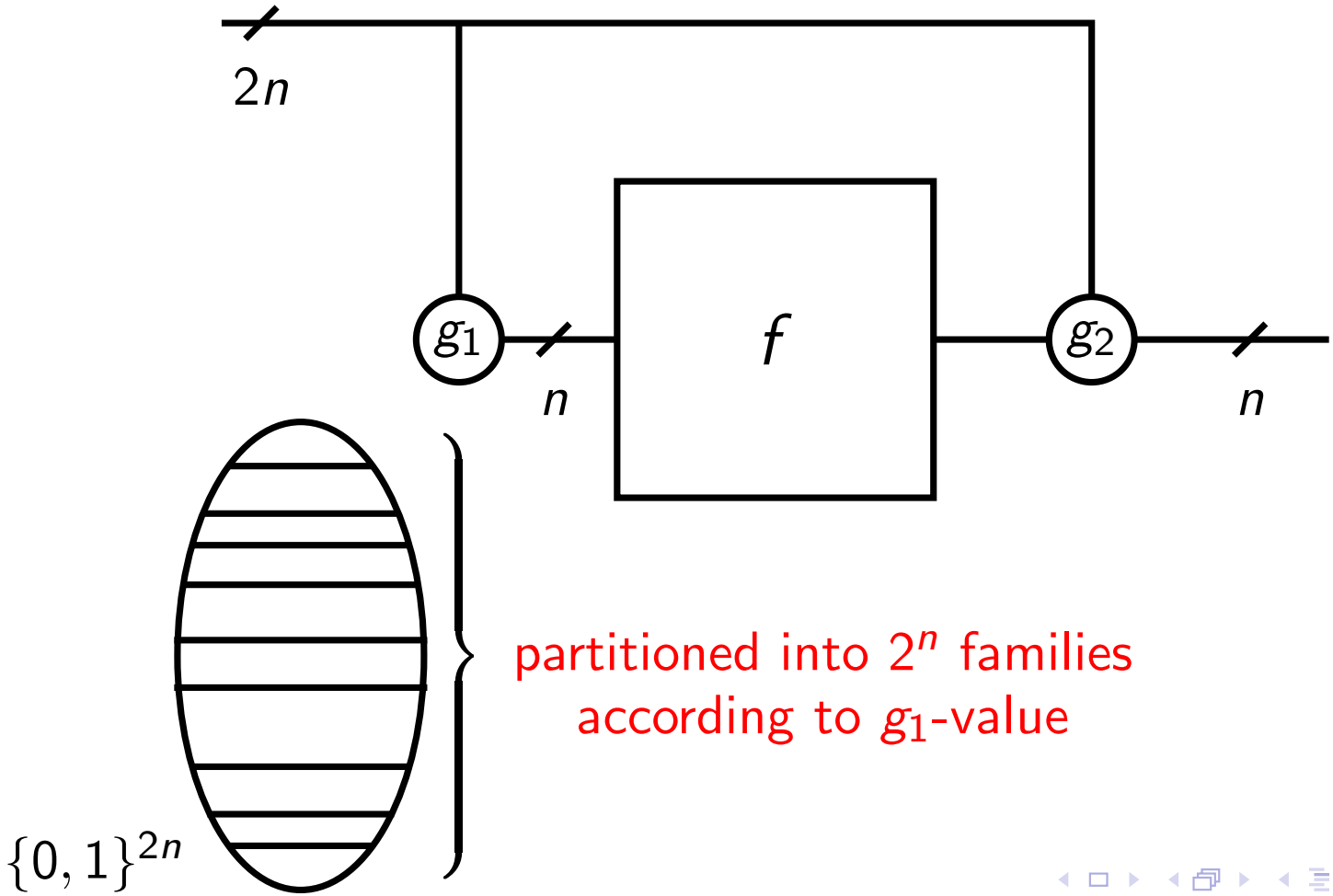

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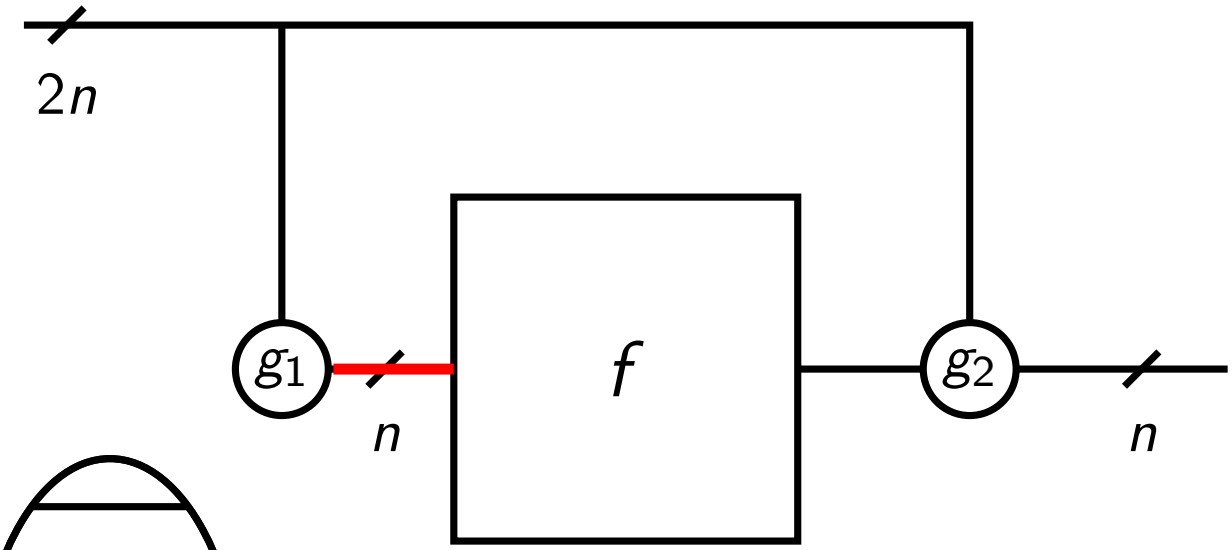

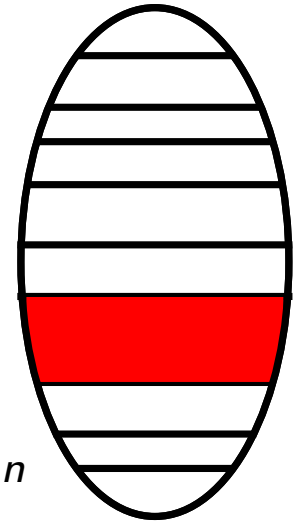

By making 1 greedy query the adversary can learn the output value for $\geq 2^n$ inputs

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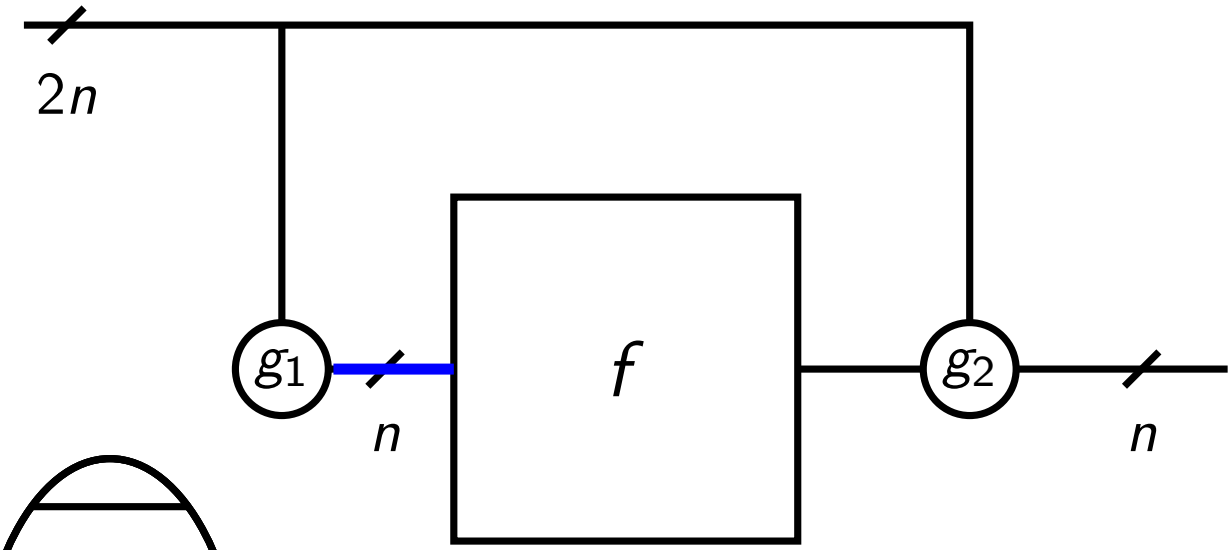

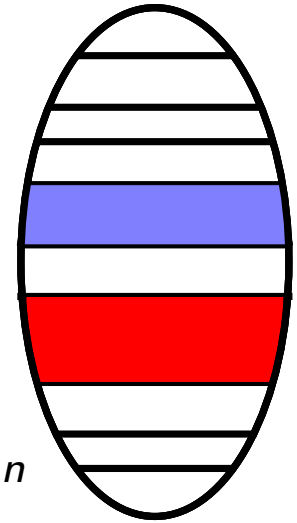

By making two greedy queries the adversary can learn the output value for $\geq 2 \cdot 2^n$ inputs

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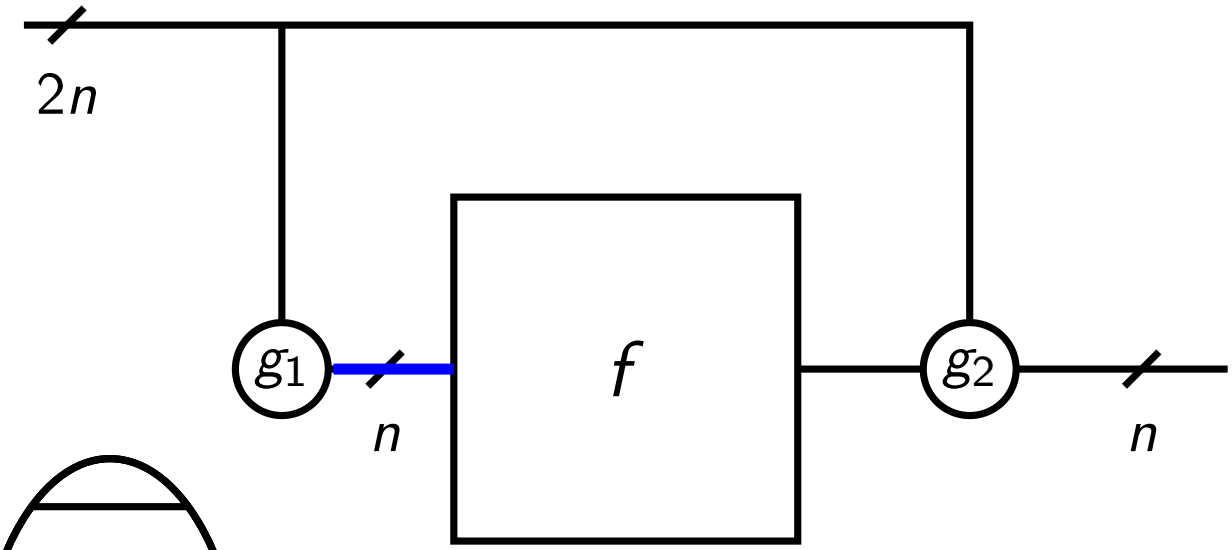

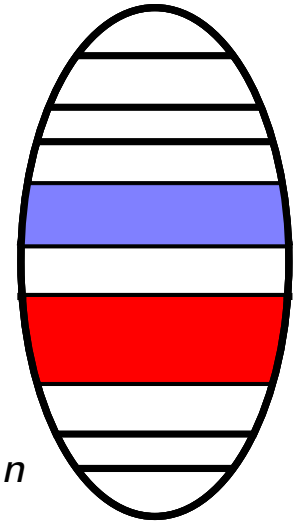

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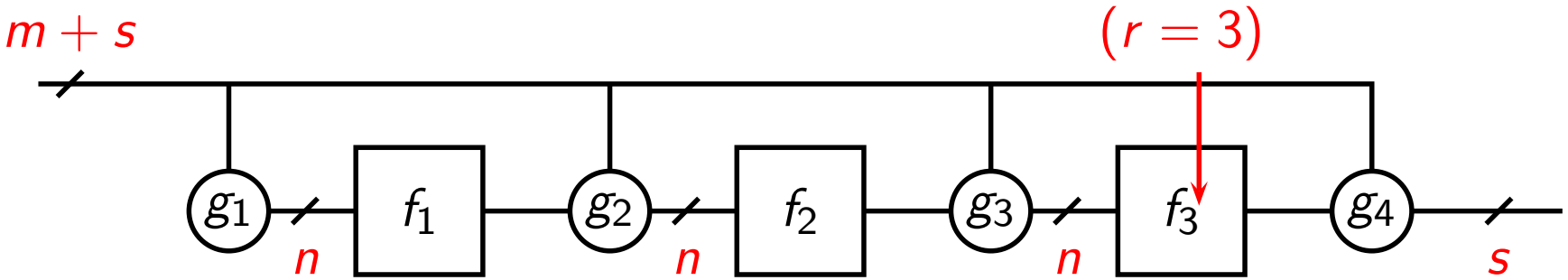
Collision security $= 2$ queries

The General Model

An $(m+s)$-bit to s-bit compression function making r calls to a primitive f of n-bit input.



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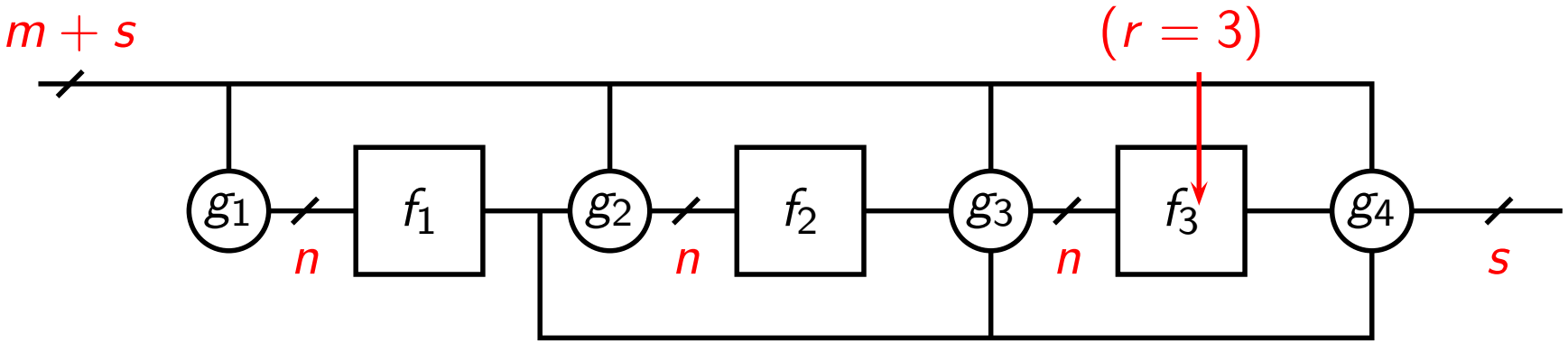

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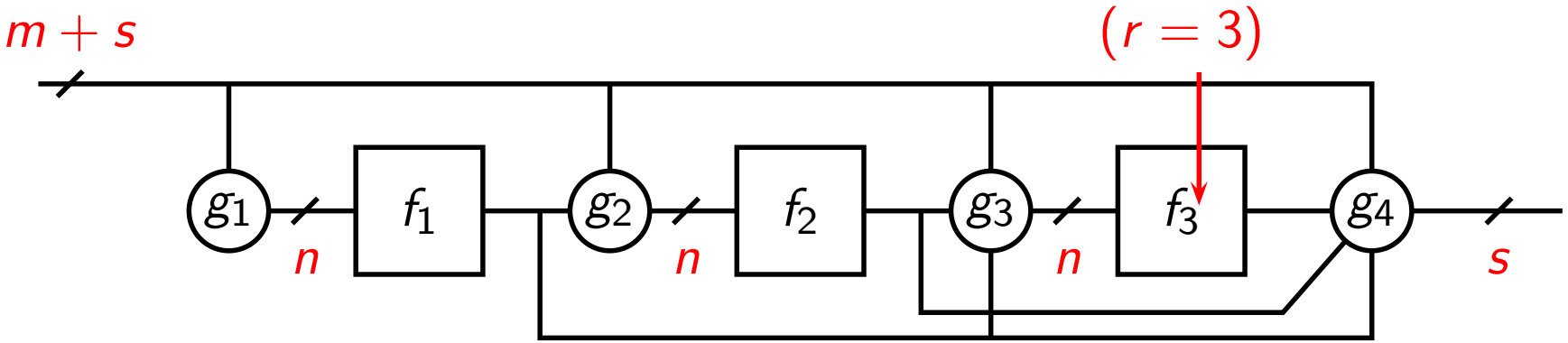

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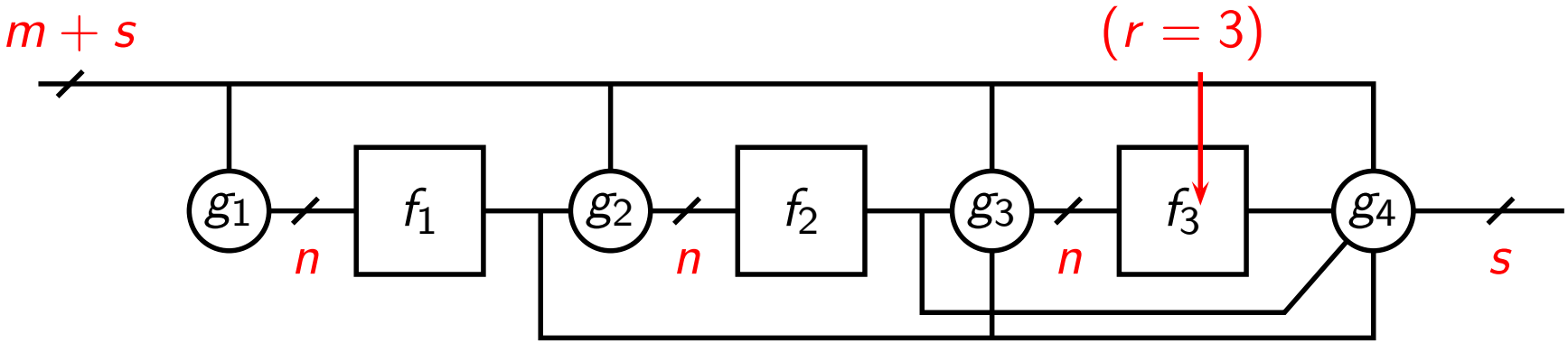

Stam's conjecture: The collision security of such a compression function is upper bounded by

$$r \cdot \min(2^{(nr-m)/(r+1)}, 2^{s/2})$$

(up to a possible constant factor).

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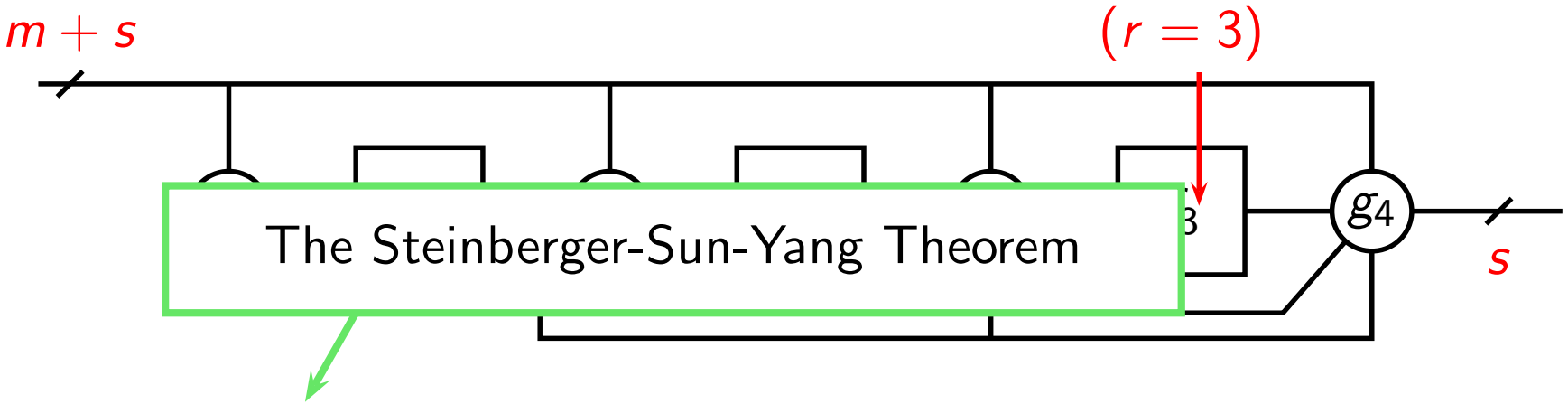

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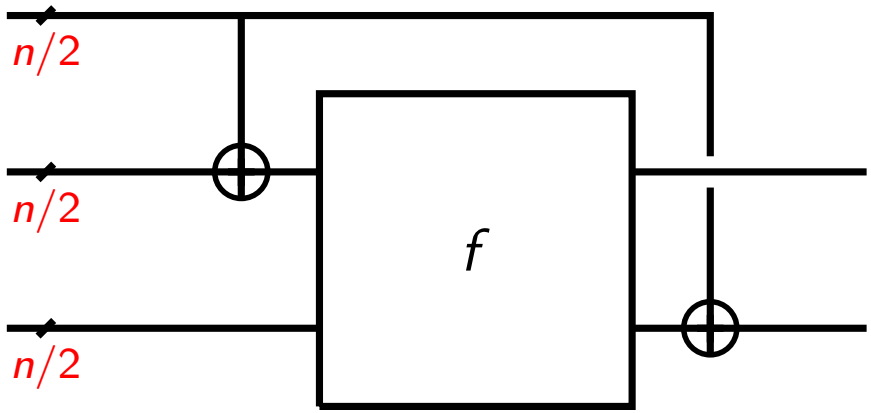

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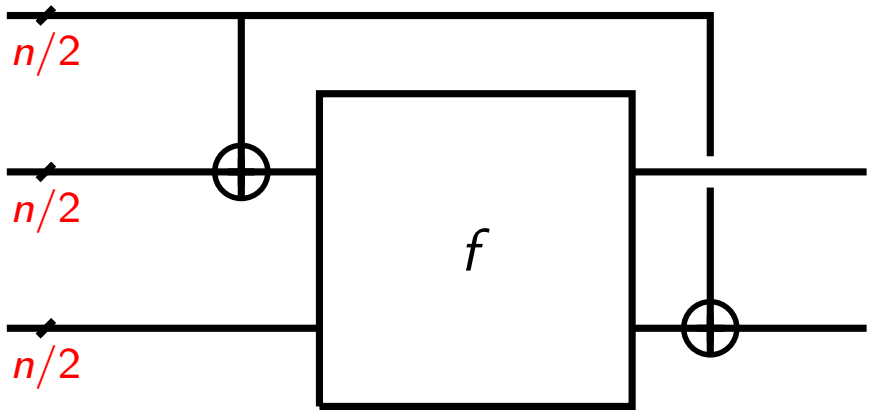

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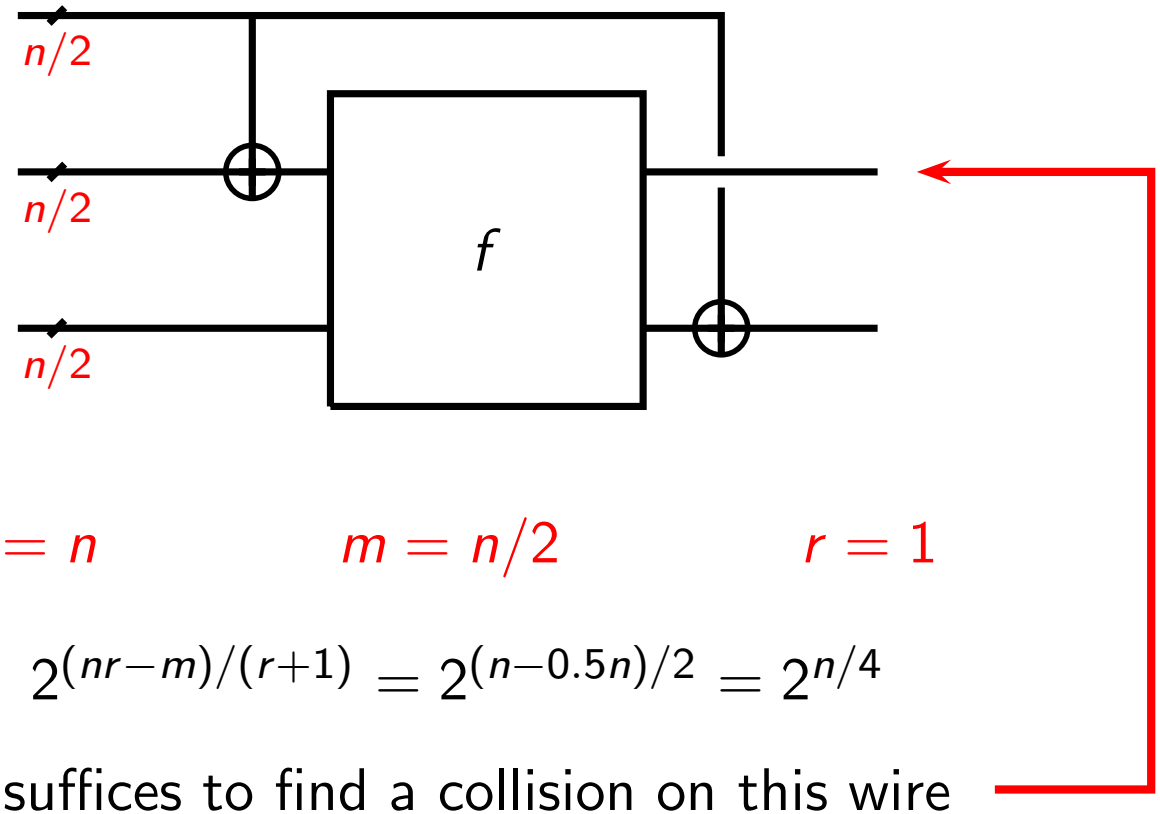

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It suffices to find a collision on this wire

(Main Theorem)

With

$$q = O(1)r2^{(nr-m)/(r+1)}$$

queries one can obtain at least

$$2^{2(s/2-(nr-m)/(r+1))}$$

collisions with high probability.

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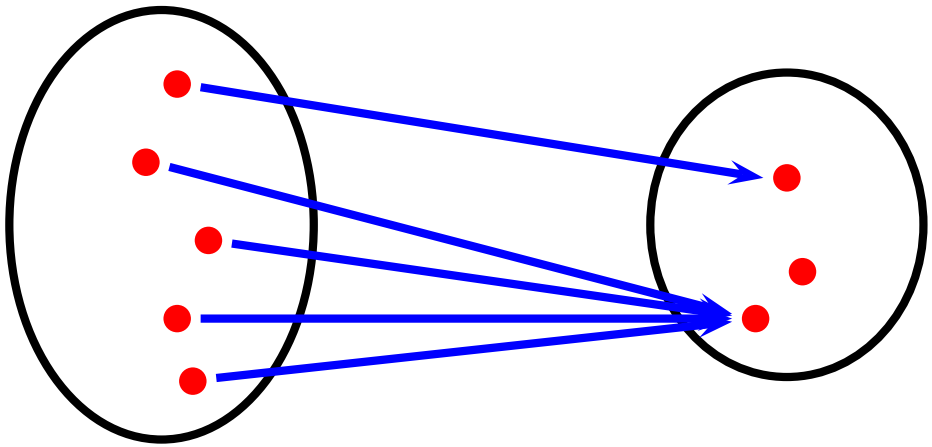

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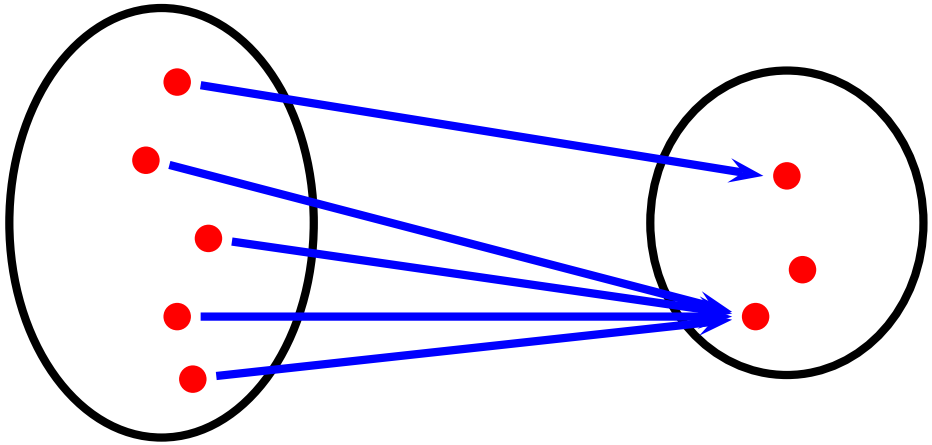

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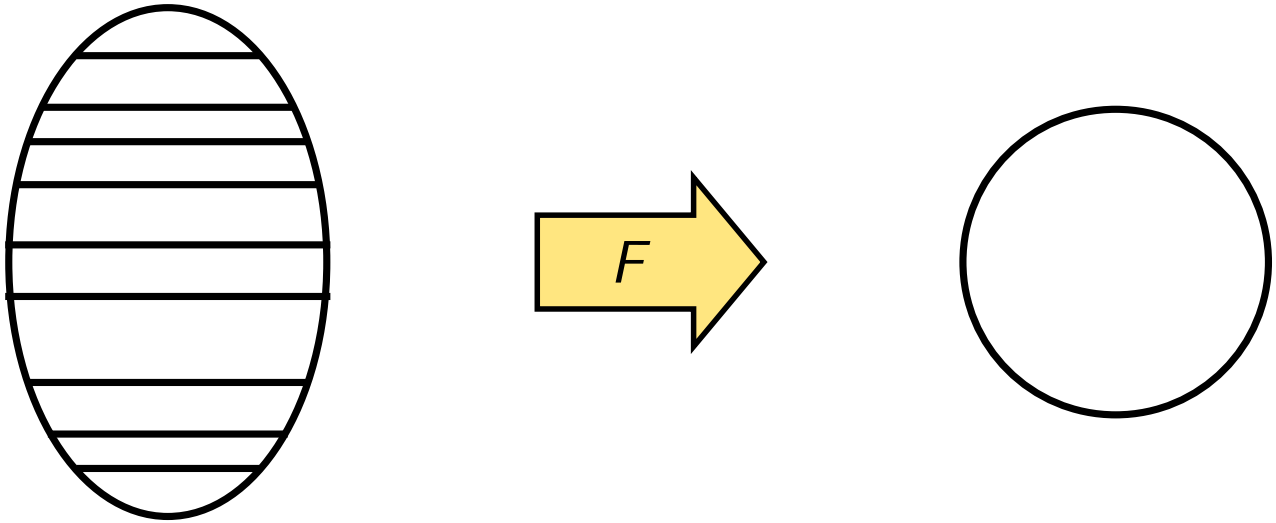
$\geq |Domain| - |Range|$

Key Lemma



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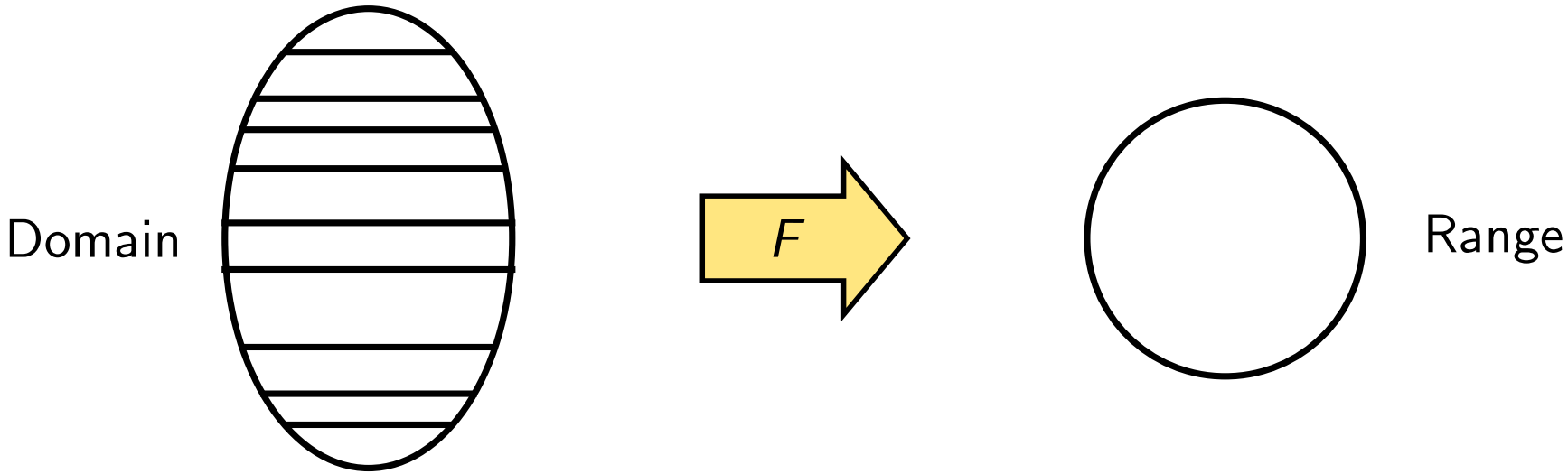

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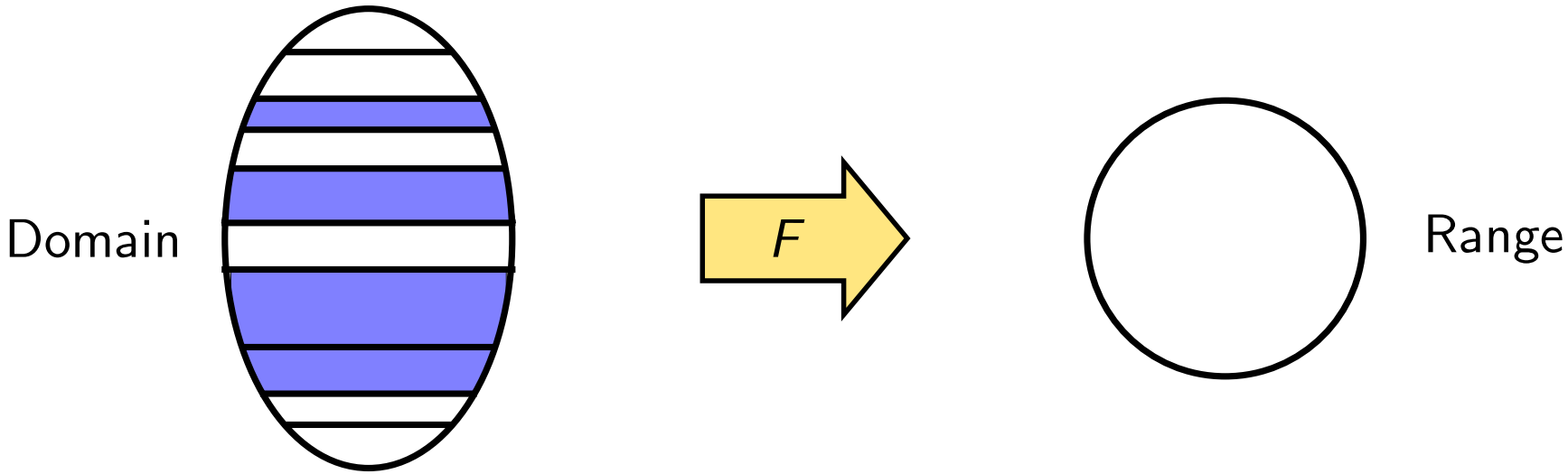
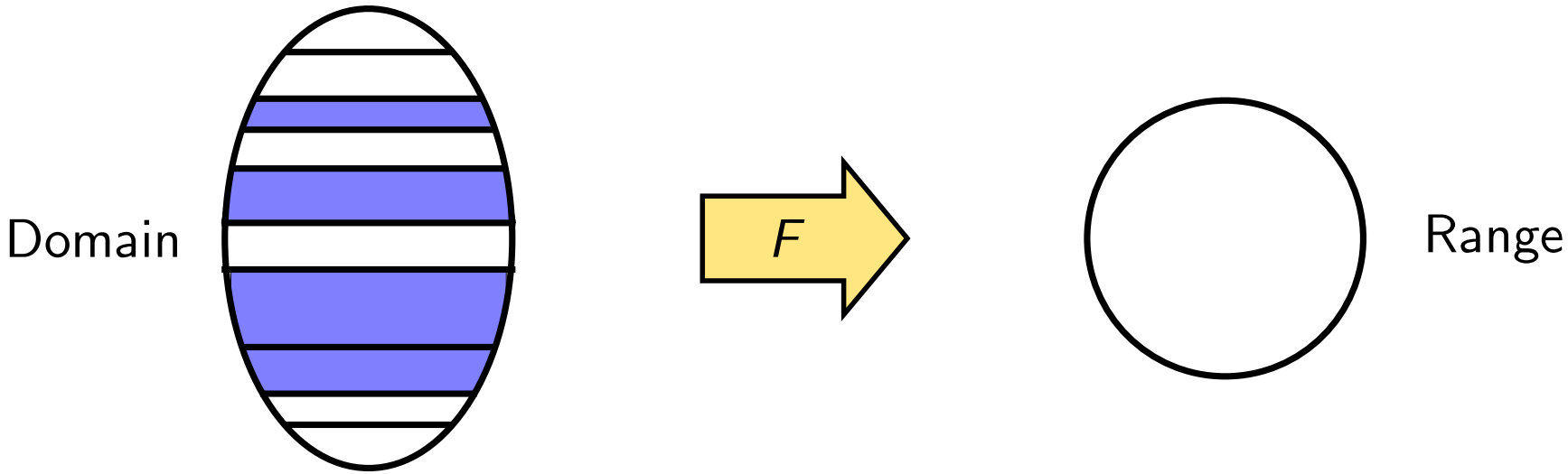


Basic idea: That if we restrict the domain by selecting a certain number of layers randomly, the number of leftover colliding inputs is close to its expectation with high probability.

Key Lemma



C: Original number of colliding inputs in F

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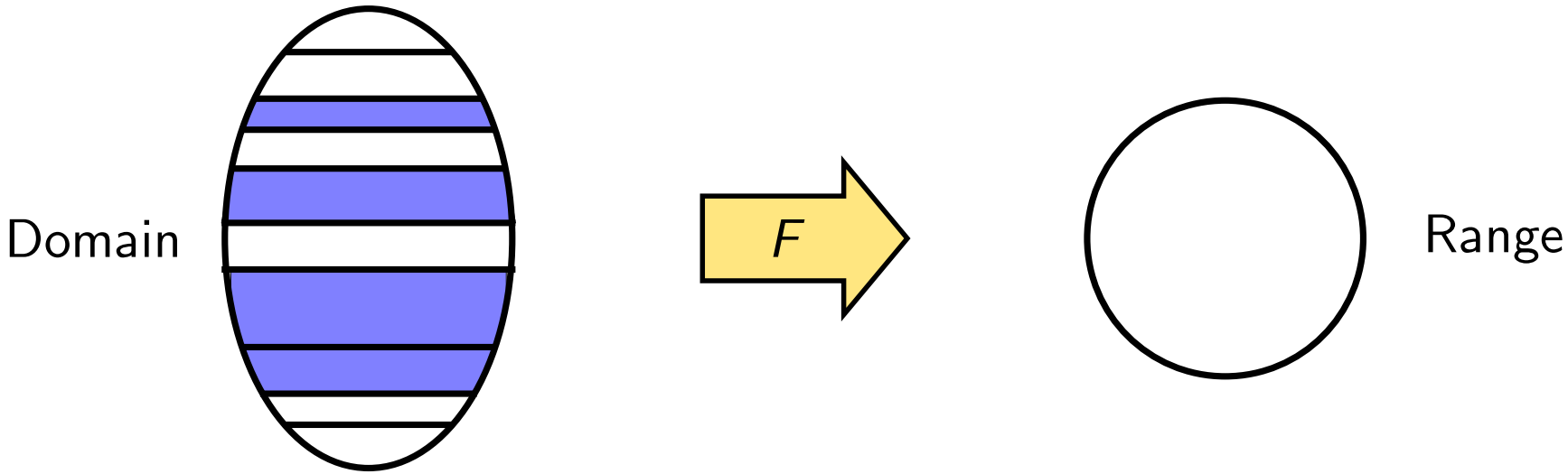

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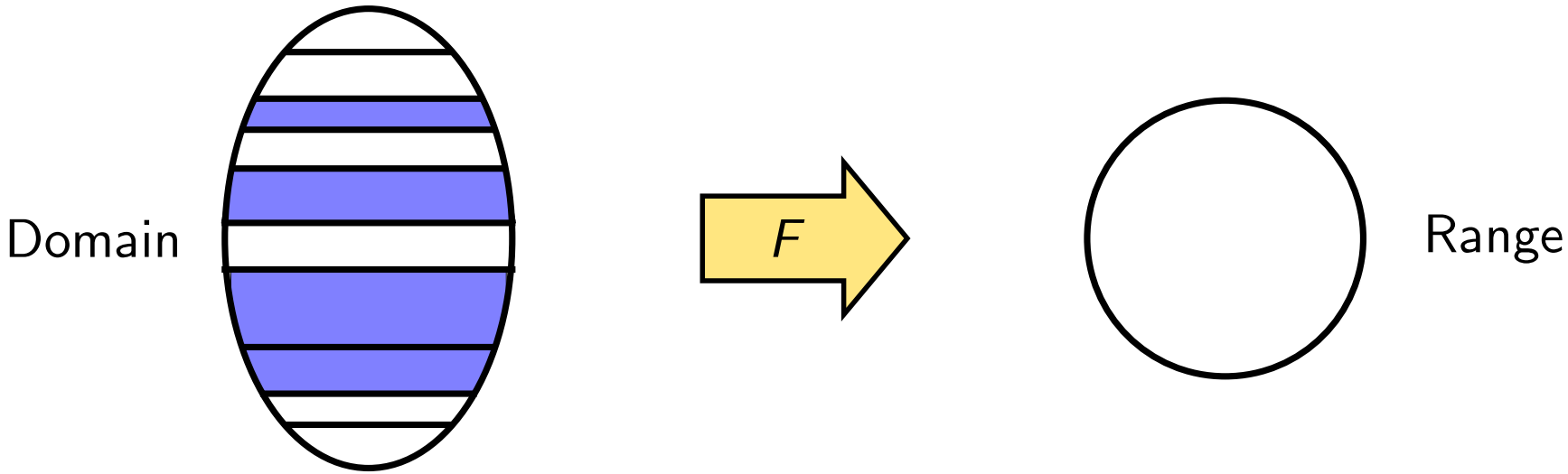

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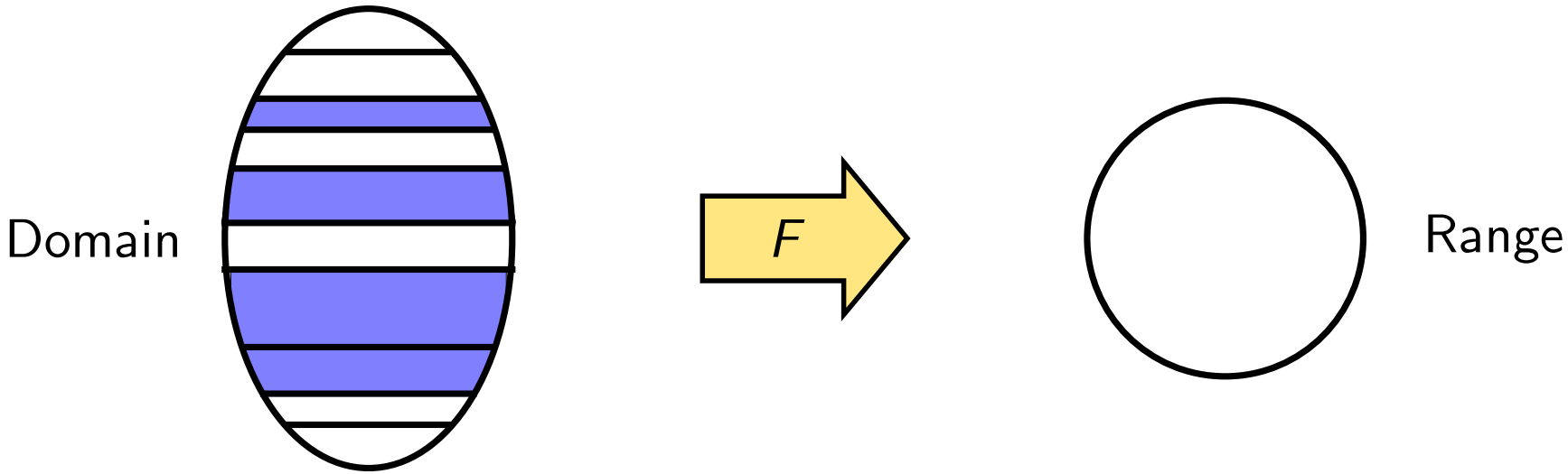

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$p = q/k$: The probability of a layer being selected

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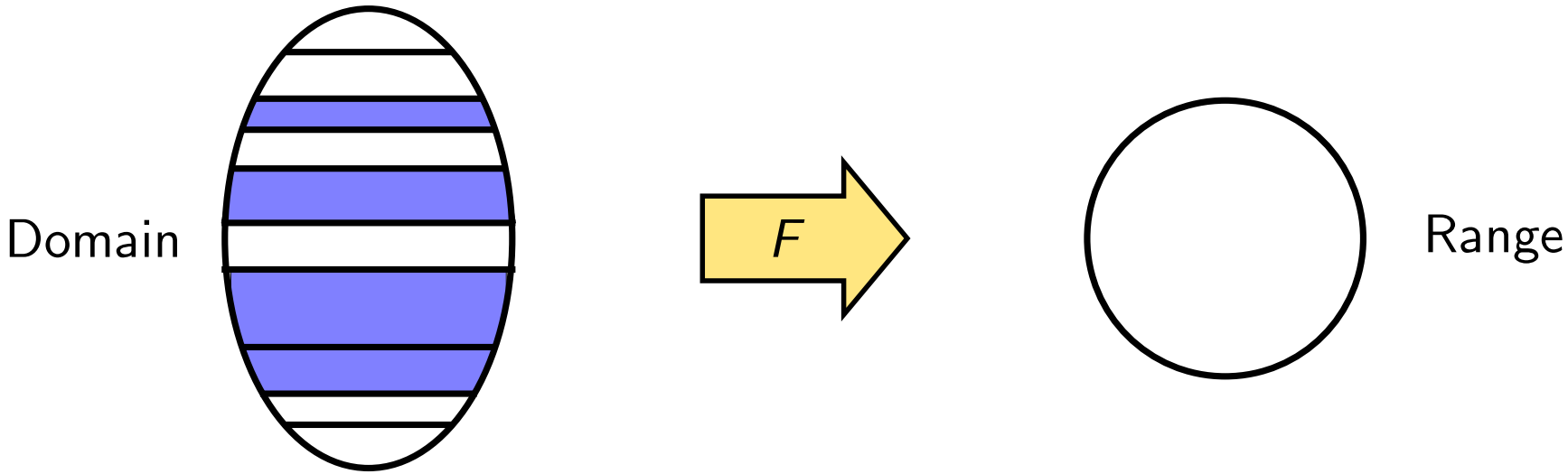

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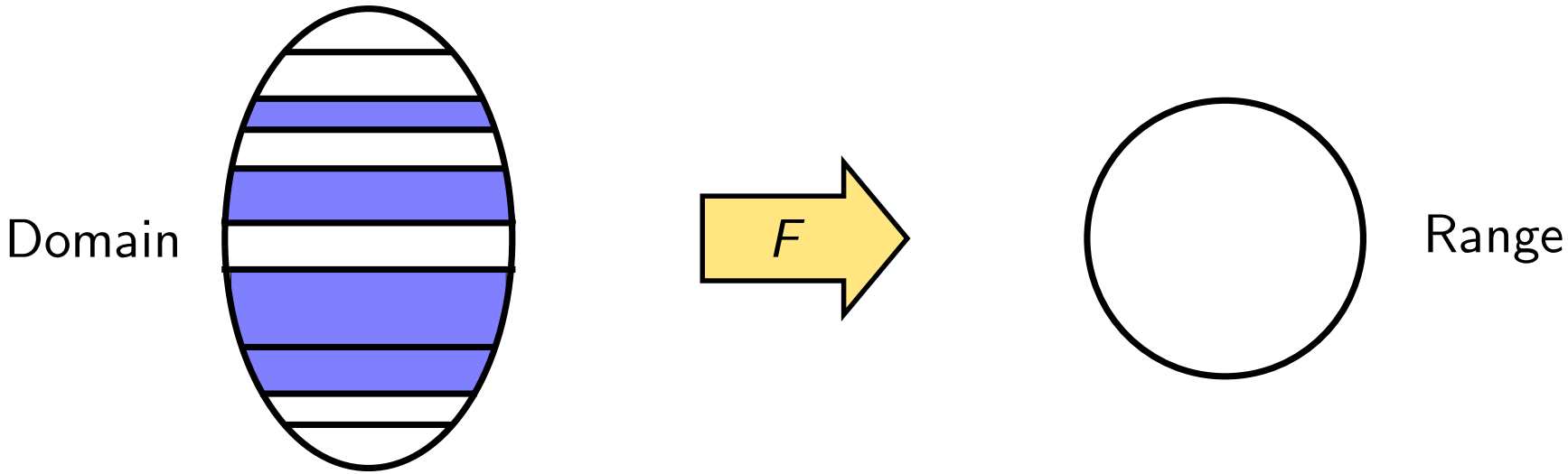

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Back-of-the-envelope expected number of leftover collisions: $p^2 C$

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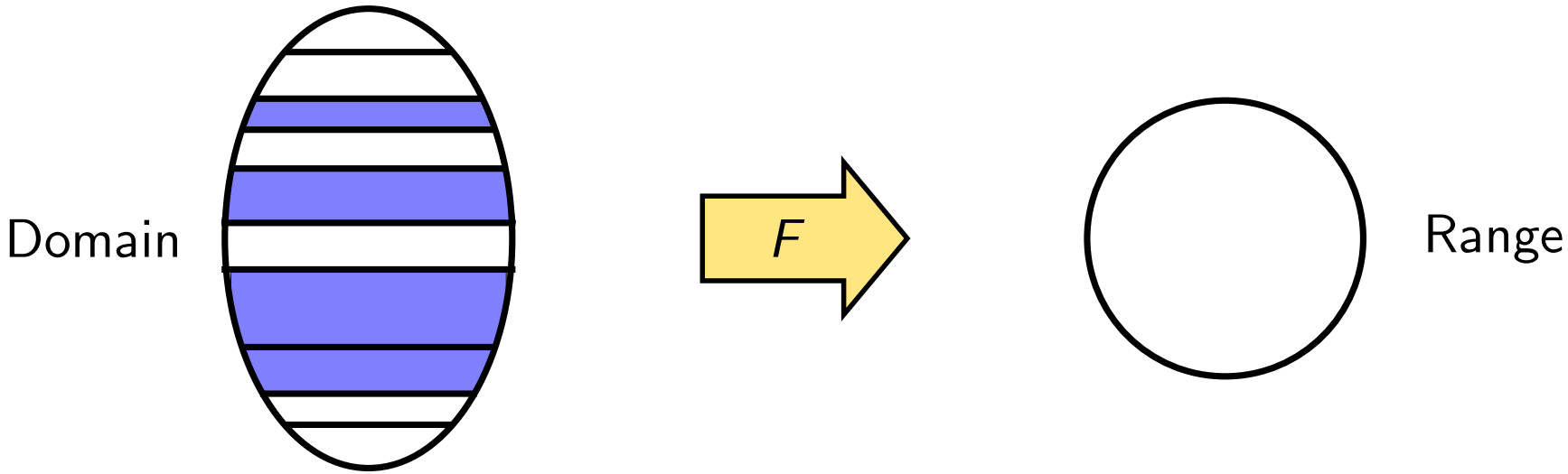

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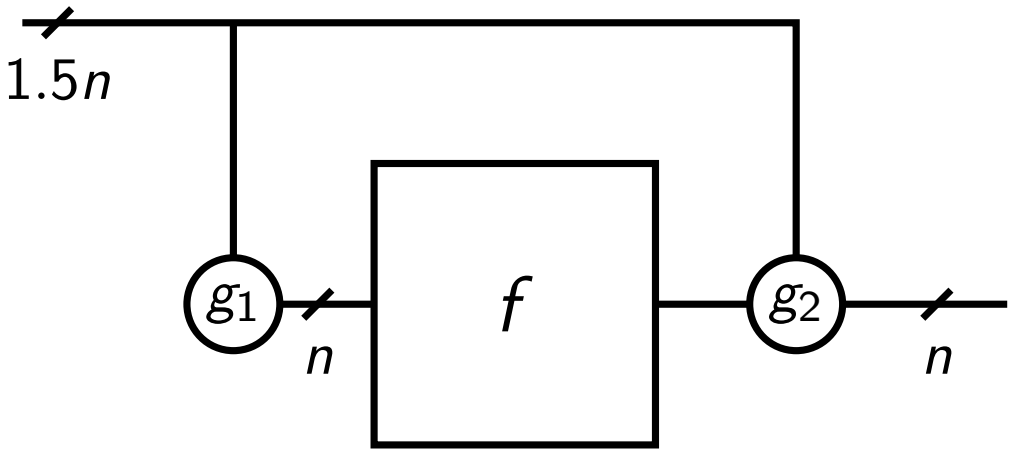
Back-of-the-envelope expected number of leftover collisions: p^2C

Lemma says: Leftover number of colliding inputs is not much less than p^2C as long as p^2C is greater than the largest size of a layer.

Proof sketch (r = 1)



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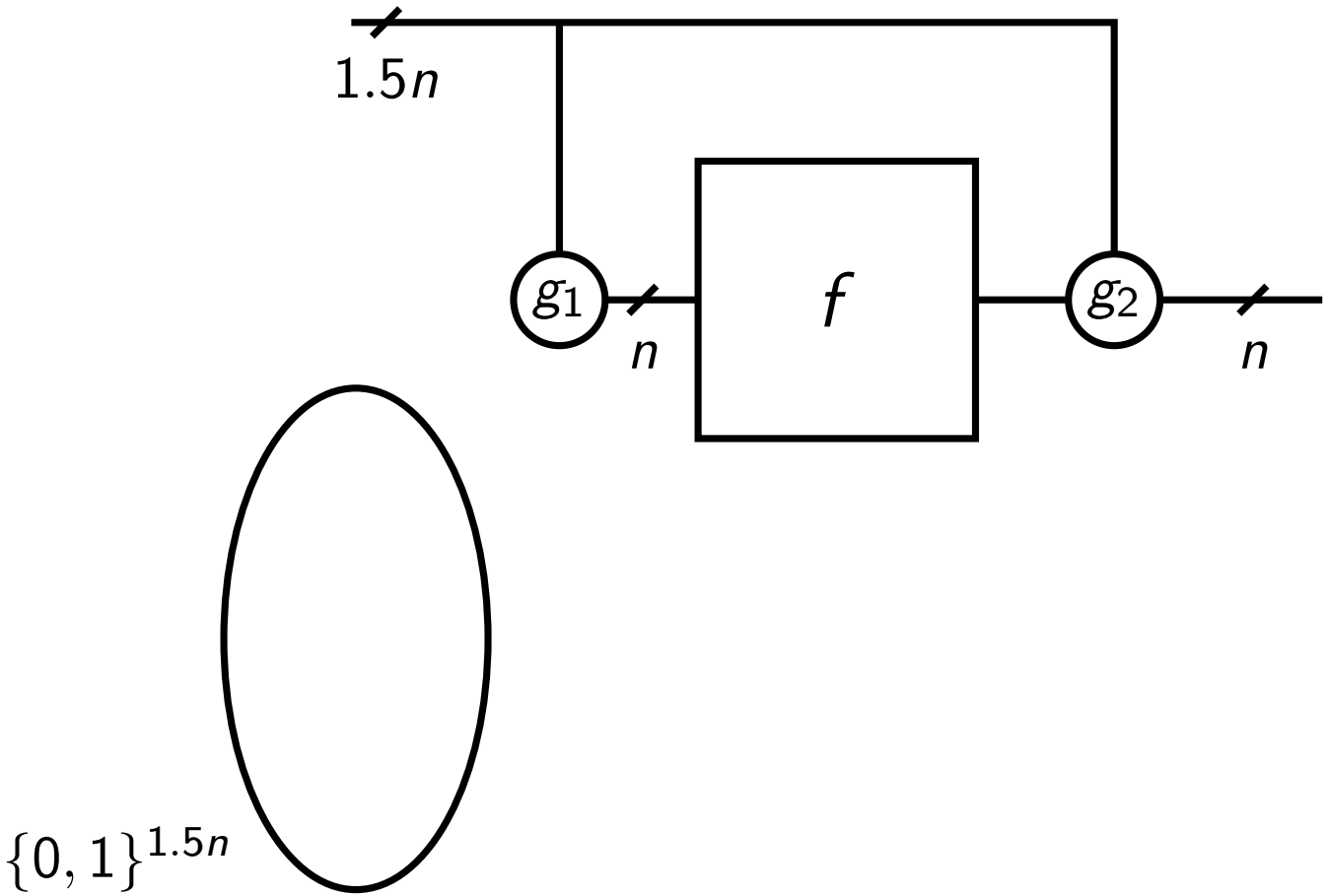

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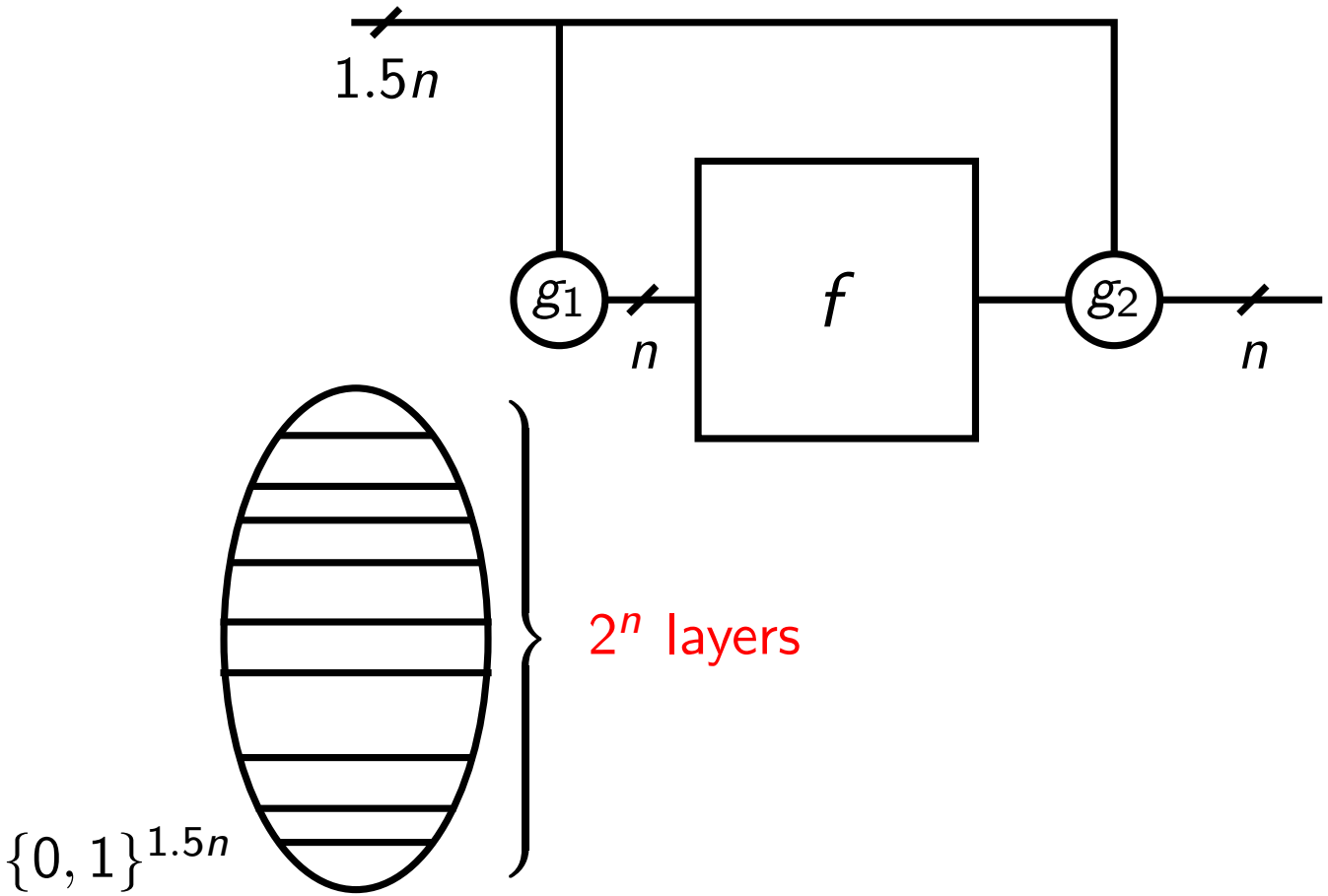

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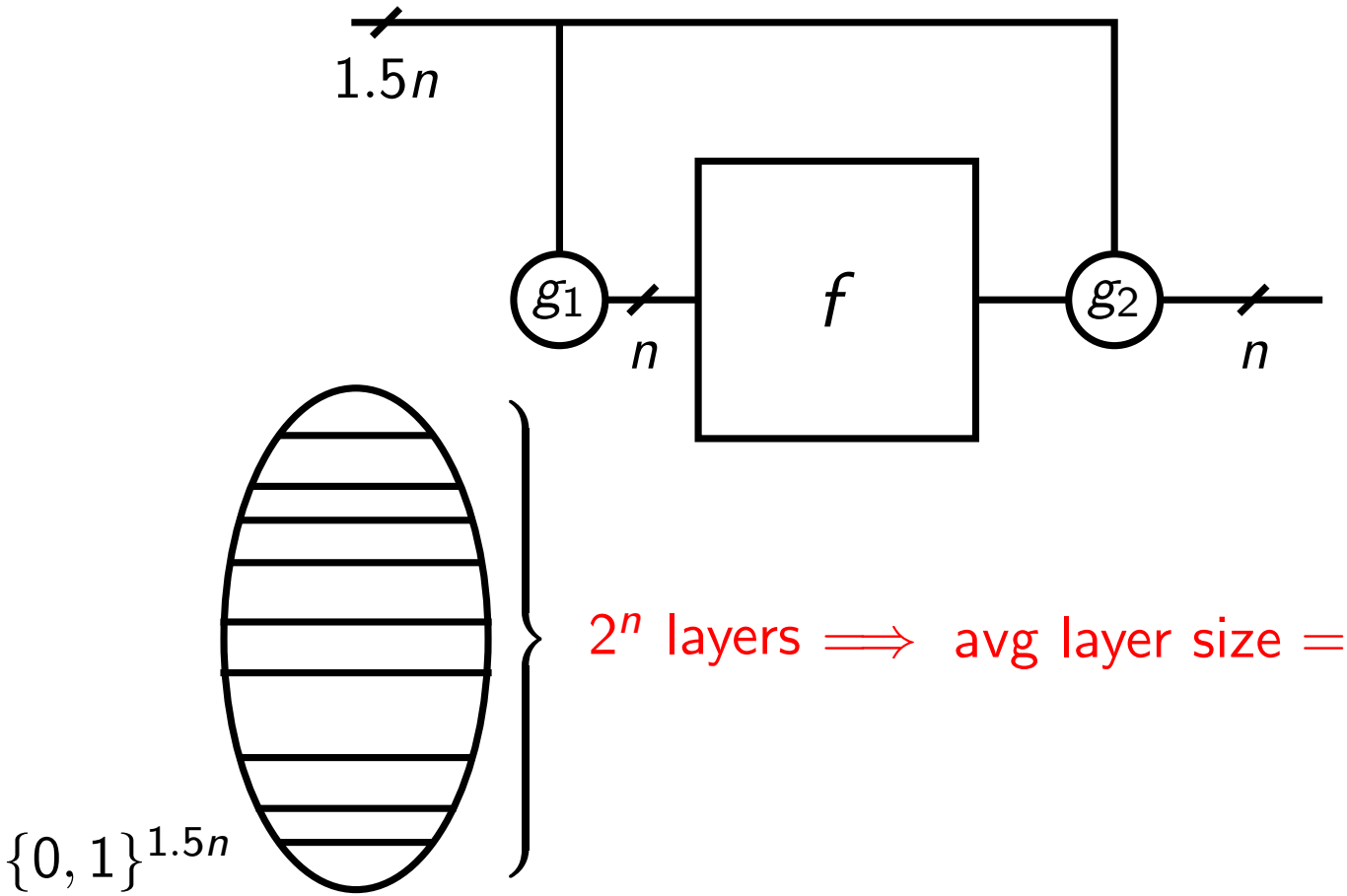

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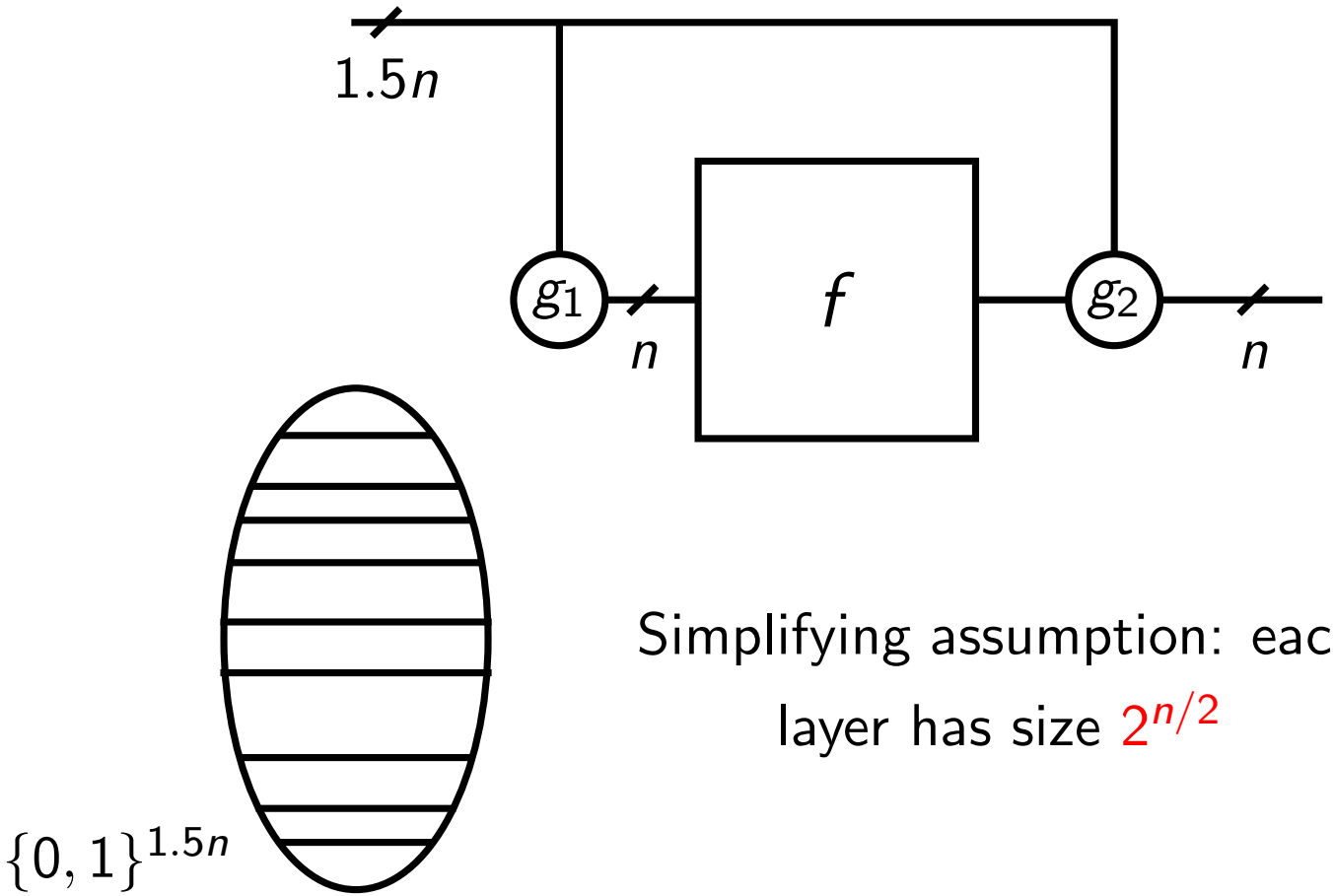

Simplifying assumption: each layer has size $2^{n/2}$

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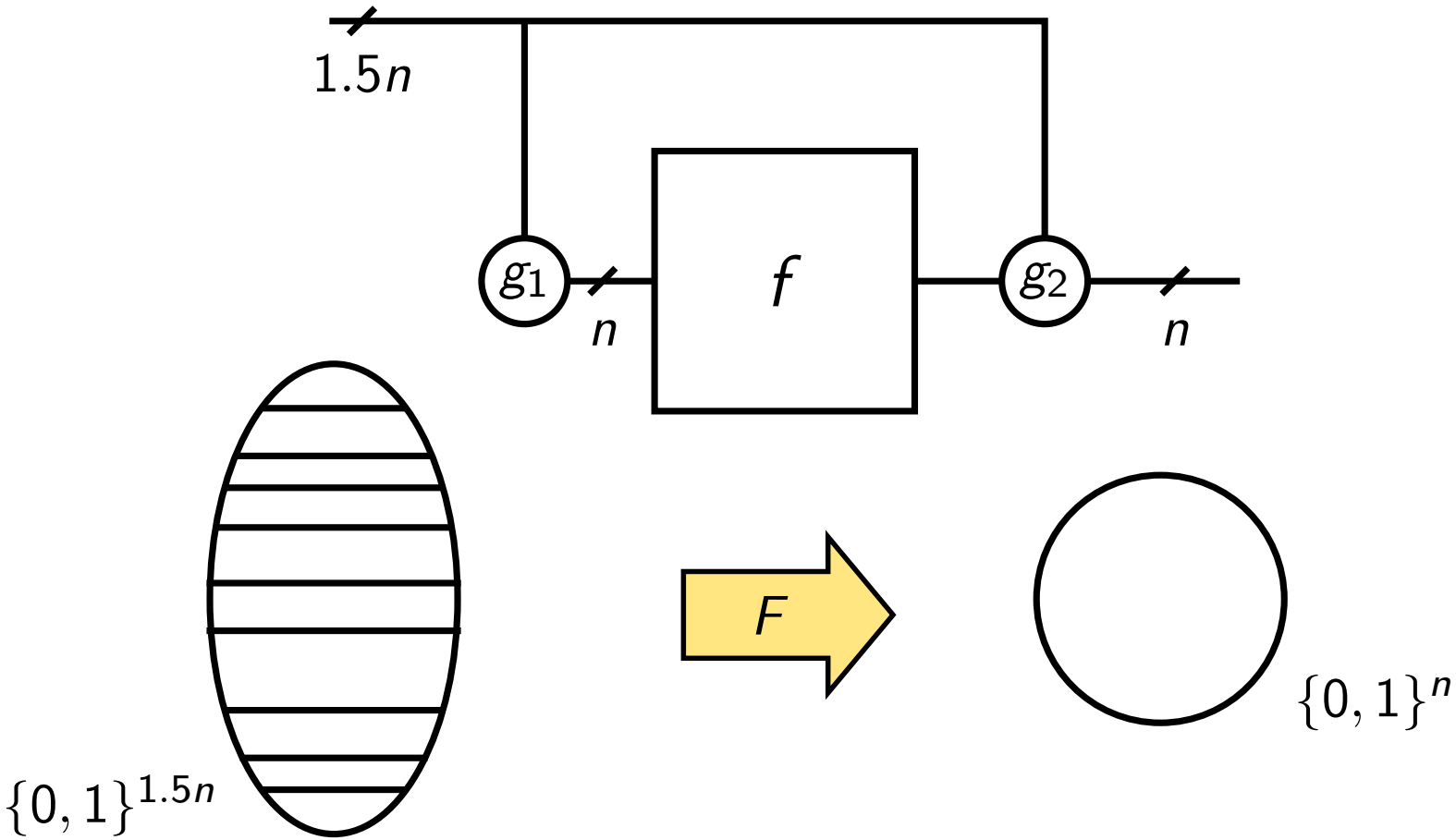

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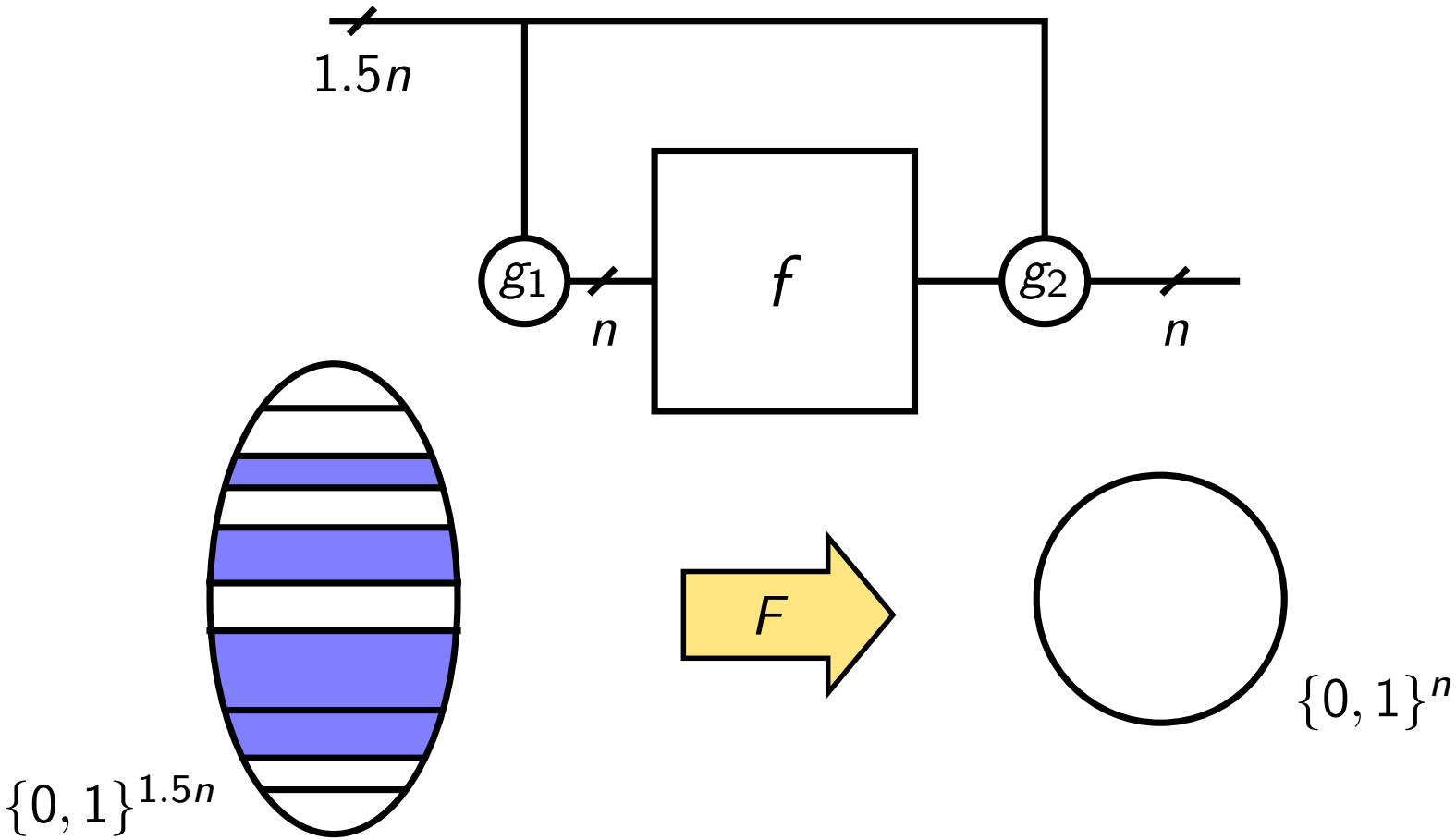

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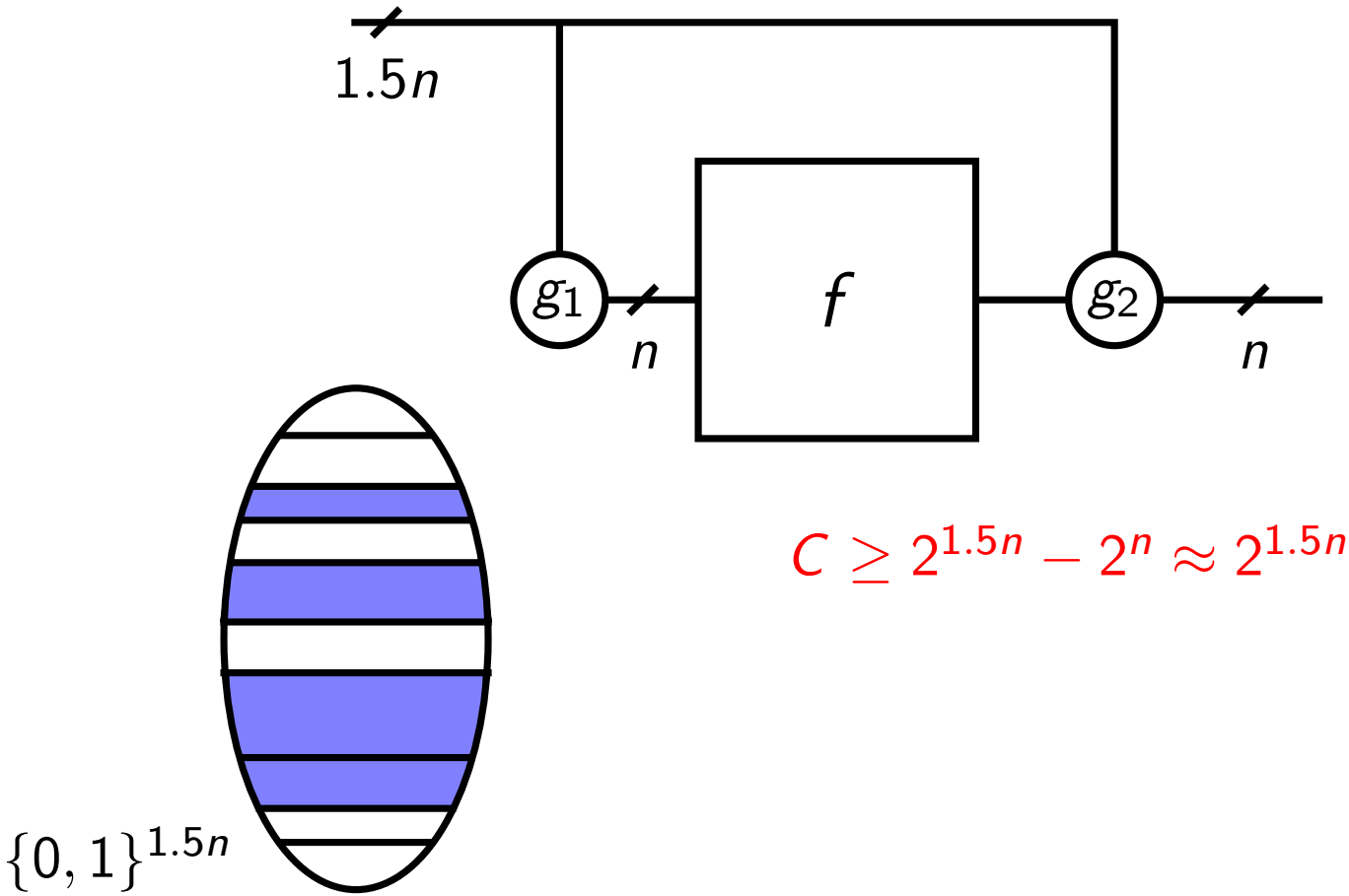

$$C \geq 2^{1.5n} - 2^n \approx 2^{1.5n}$$

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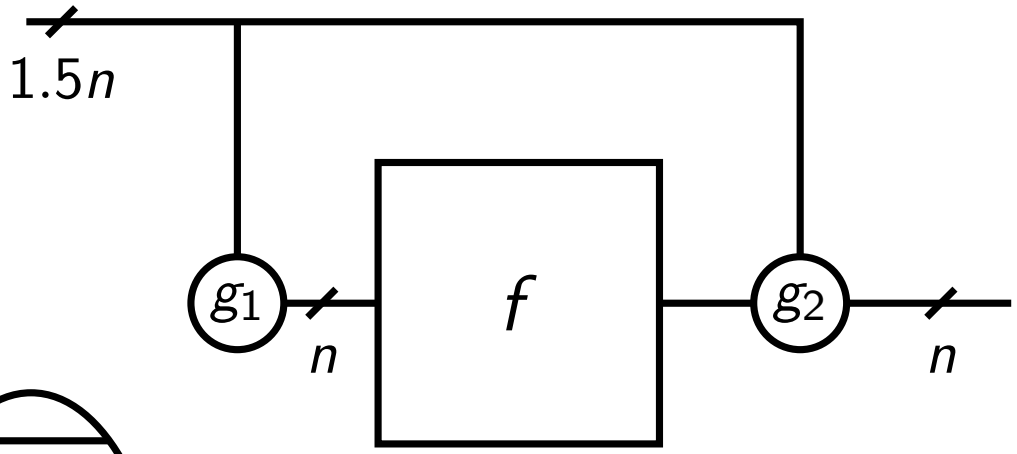
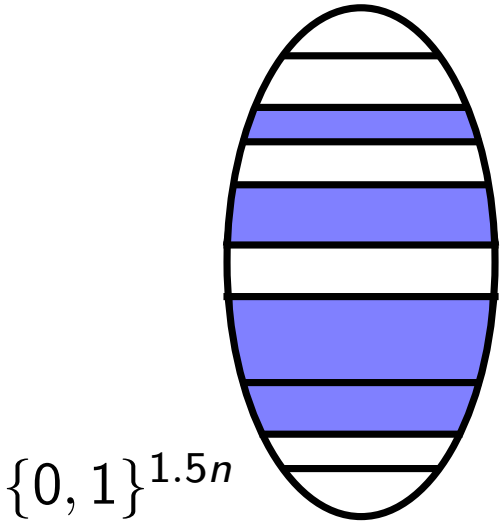
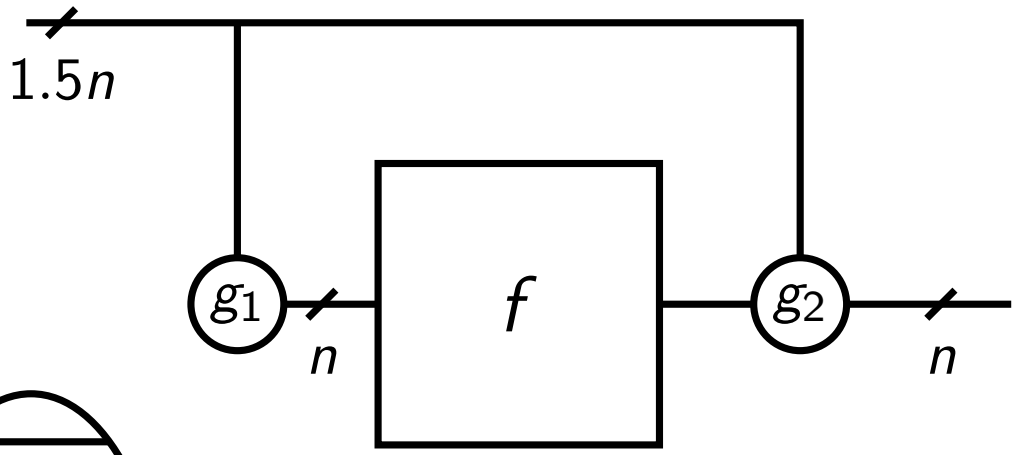


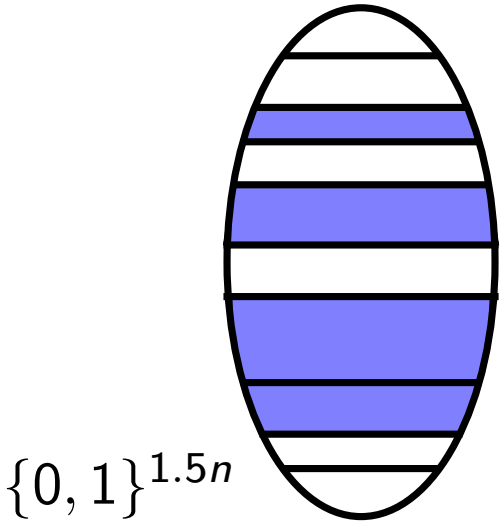


$$C \geq 2^{1.5n} - 2^n \approx 2^{1.5n}$$

$$k = 2^n,\ q = 2^{n/4},\ p = q/k = 2^{-3n/4}$$

Proof sketch (r = 1)





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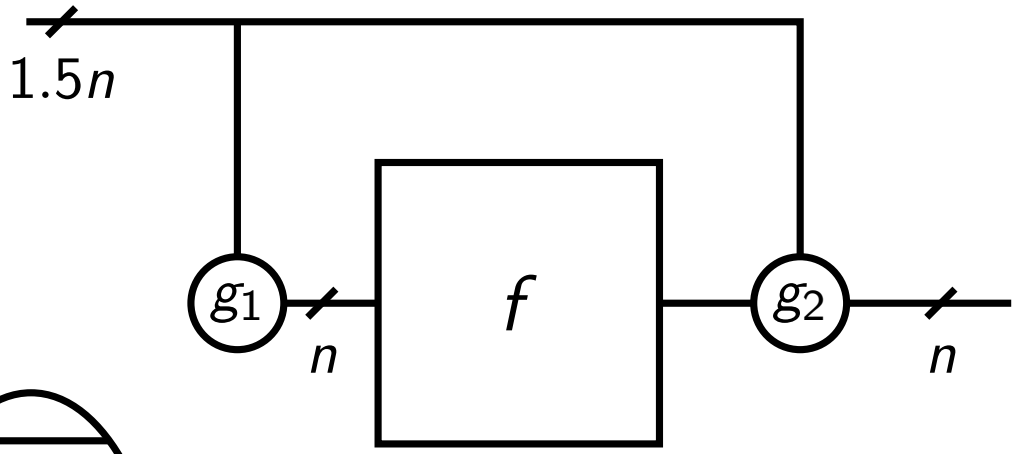
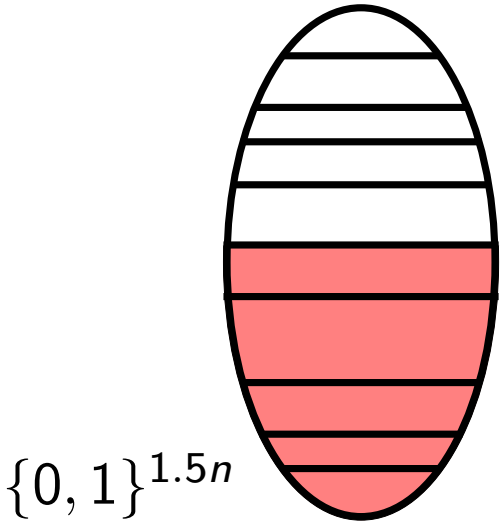
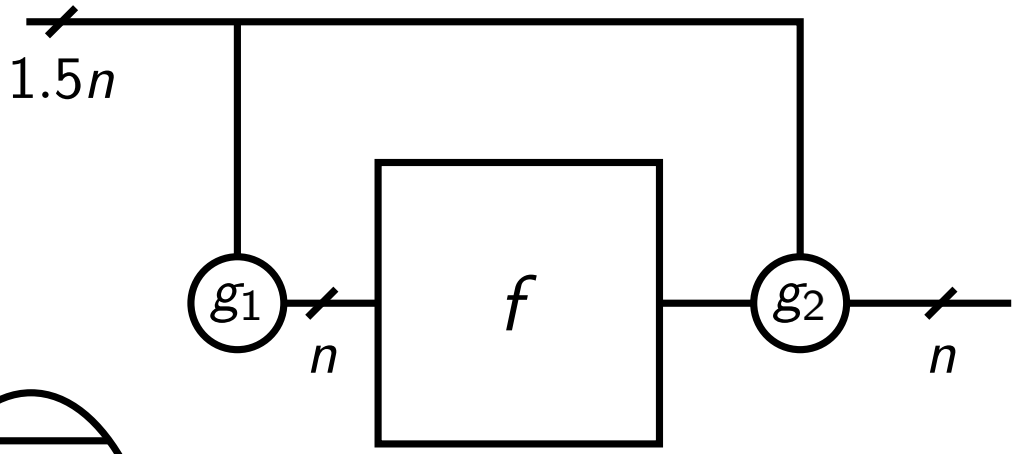
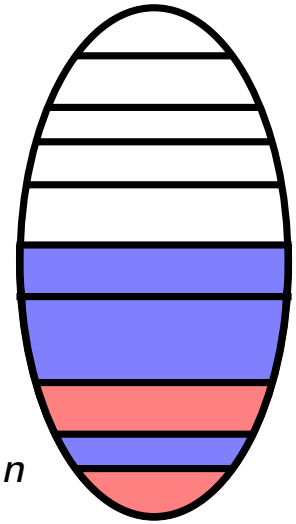




Idea: Pre-select a "ground set" of $2^{n/2+1}$ layers

Proof sketch (r = 1)

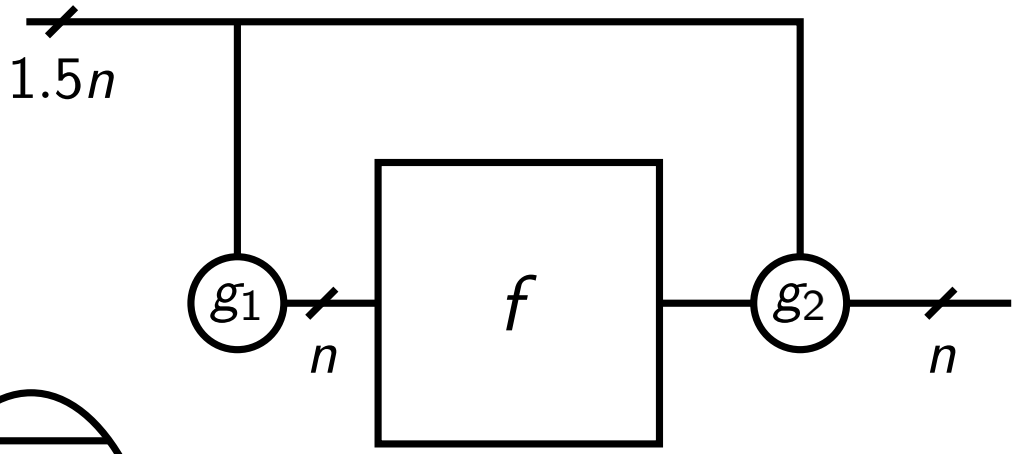


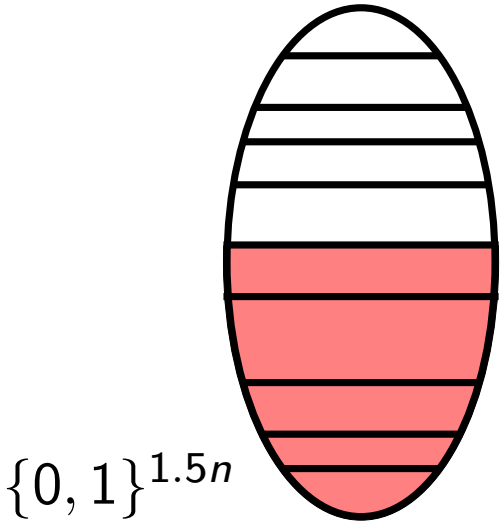


Idea: Pre-select a "ground set" of $2^{n/2+1}$ layers

...then select q layers among those...

Proof sketch (r = 1)

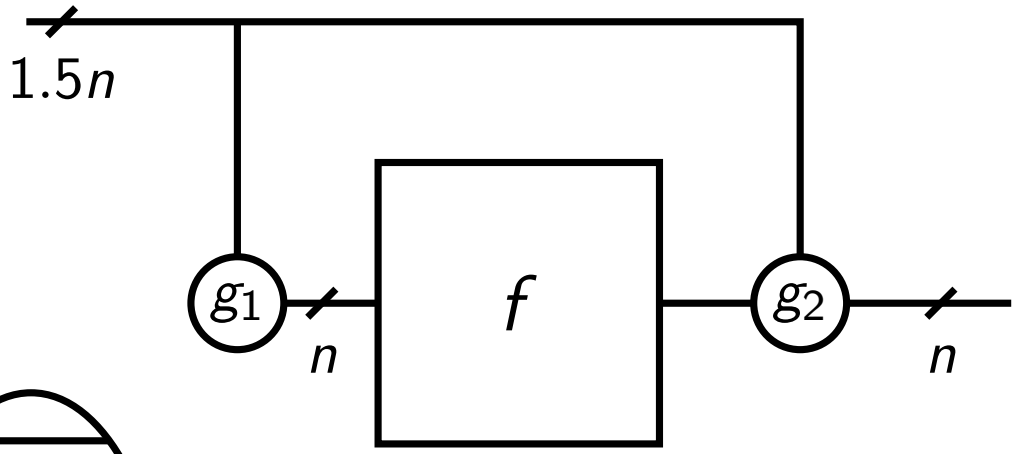


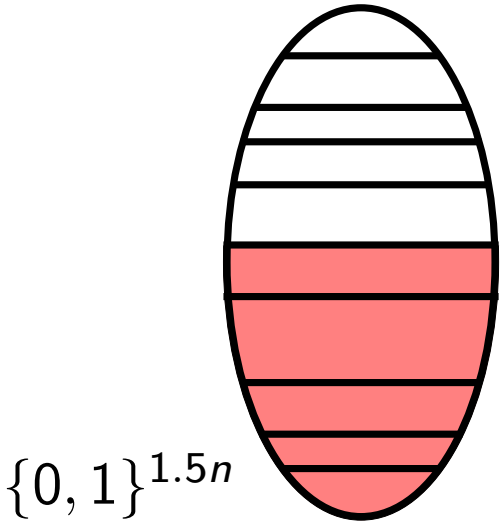


Idea: Pre-select a "ground set" of $2^{n/2+1}$ layers

Have $|Domain| = 2^{n/2+1} \cdot 2^{n/2} = 2^{n+1}$

Proof sketch (r = 1)



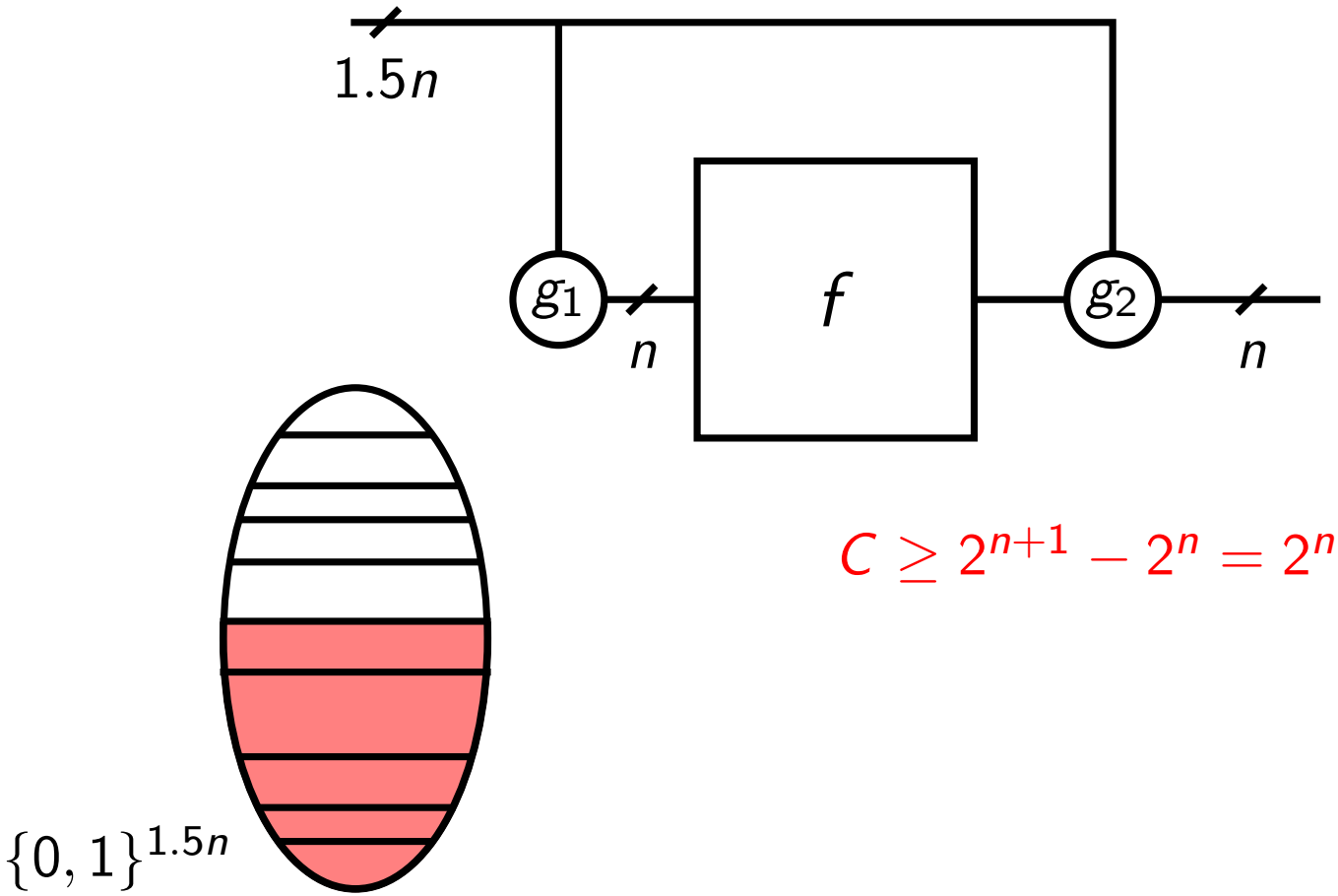


Idea: Pre-select a "ground set" of $2^{n/2+1}$ layers

Have $|Domain| = 2^{n/2+1} \cdot 2^{n/2} = 2^{n+1}$

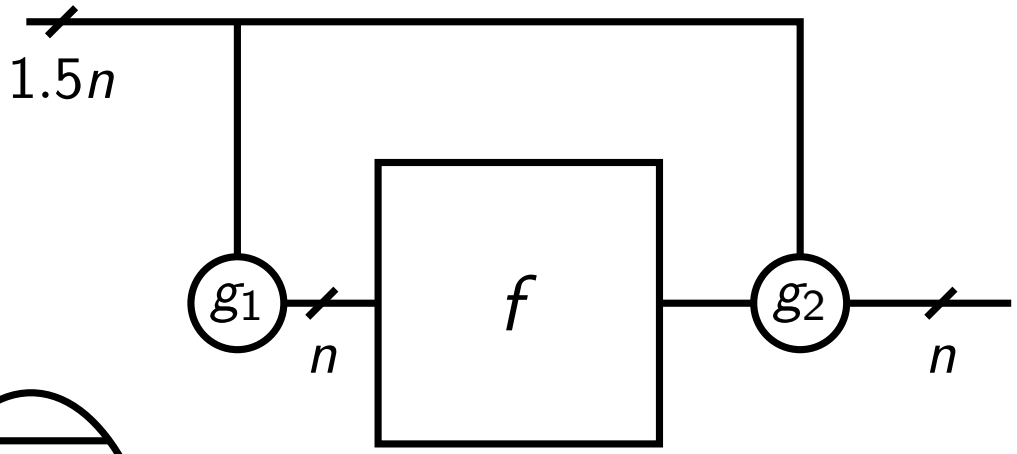
...and $|Range| = 2^n$, so still have 2^n collisions

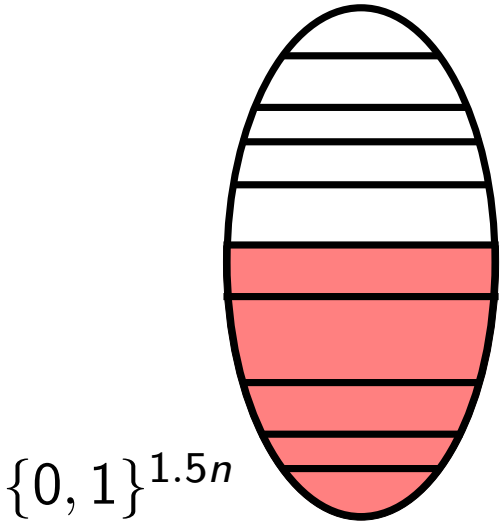
Proof sketch (r = 1)



$$C \geq 2^{n+1} - 2^n = 2^n$$

Proof sketch (r = 1)





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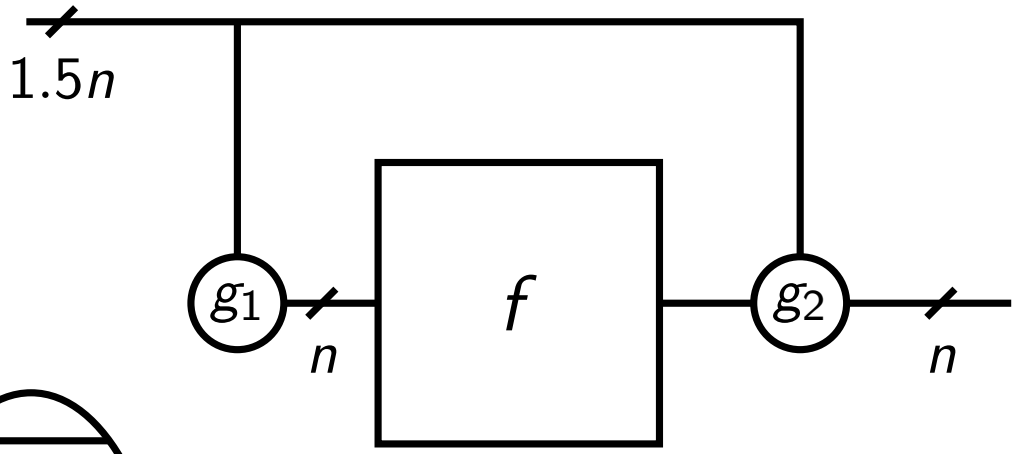

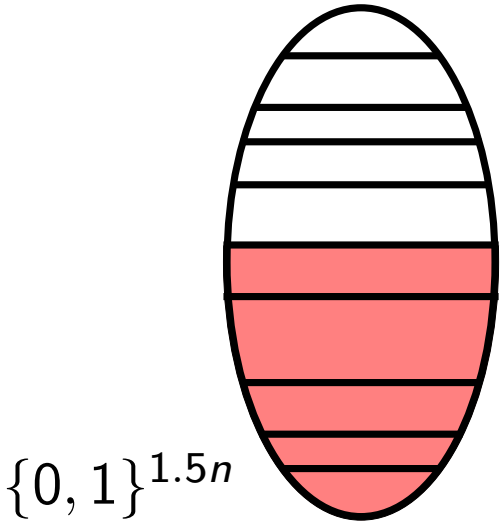

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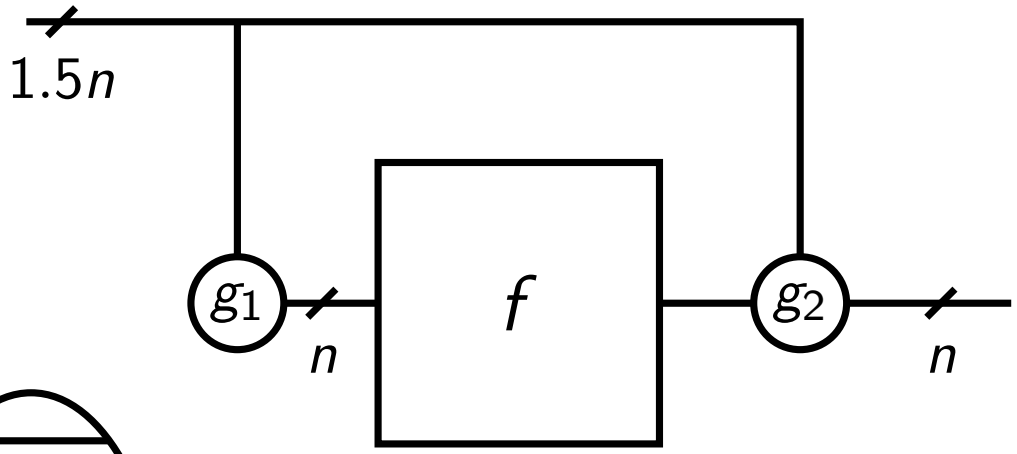

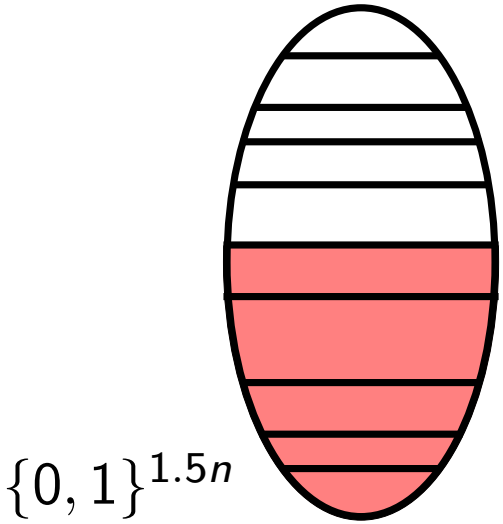

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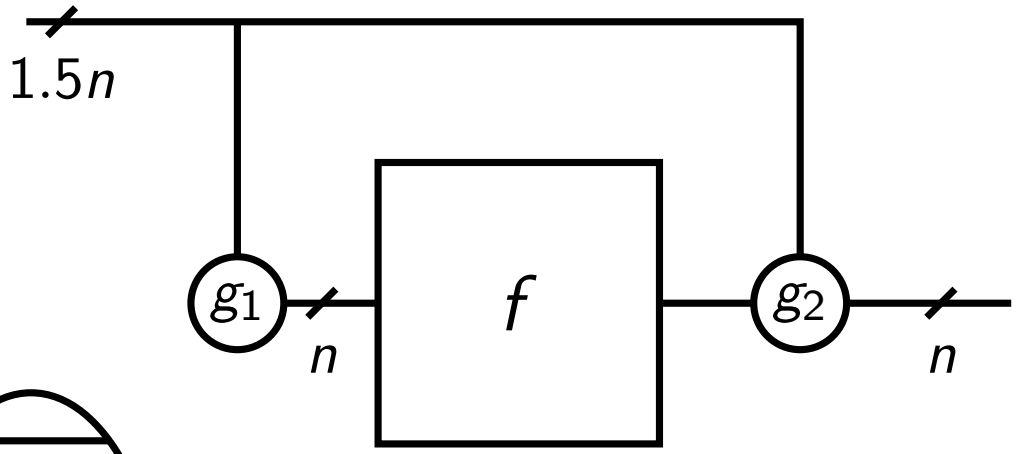

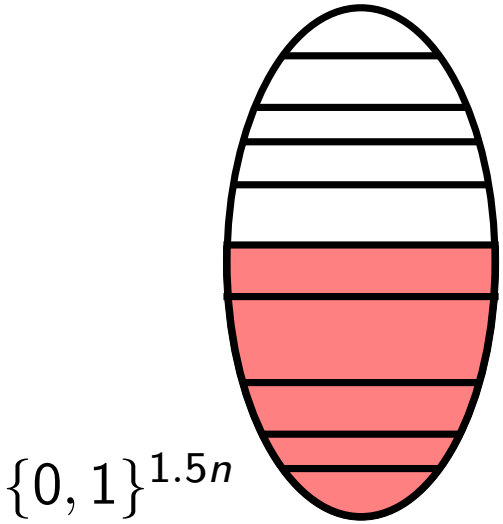

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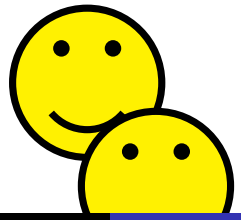

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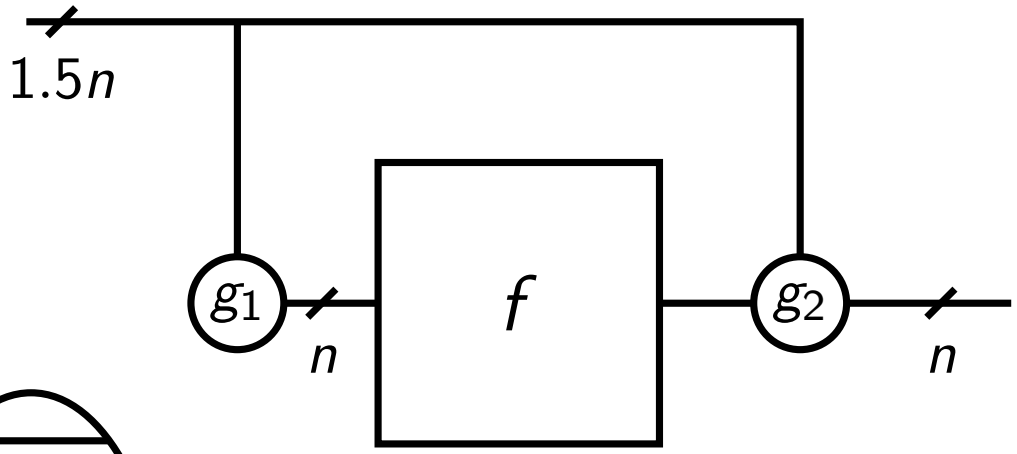

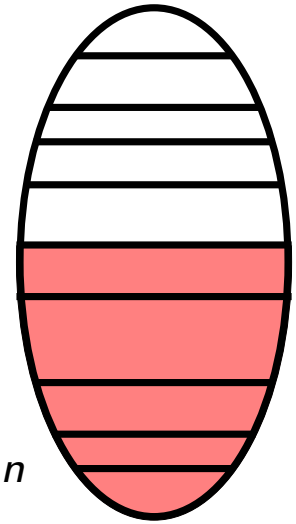

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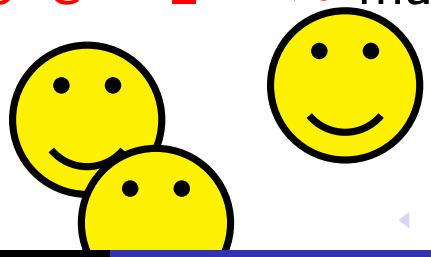

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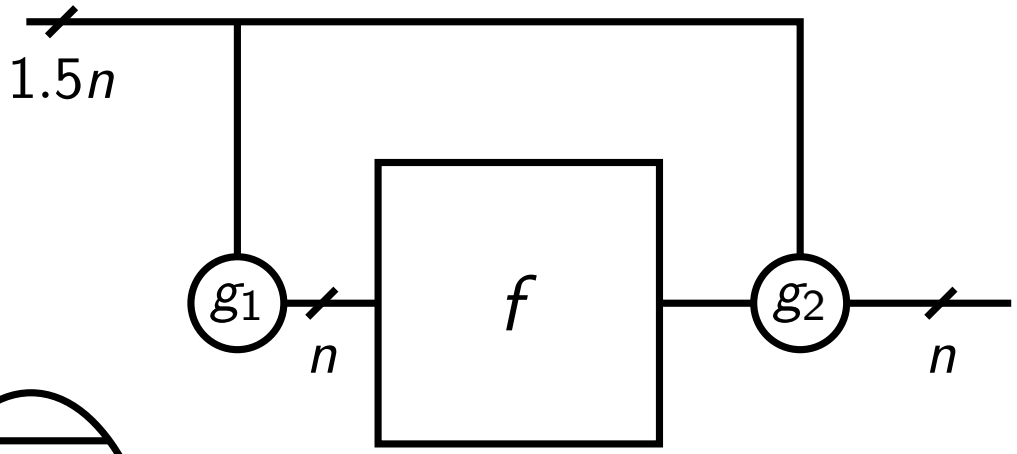

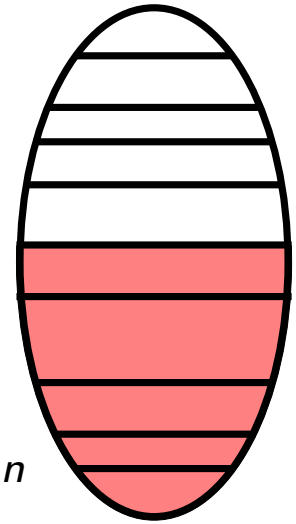

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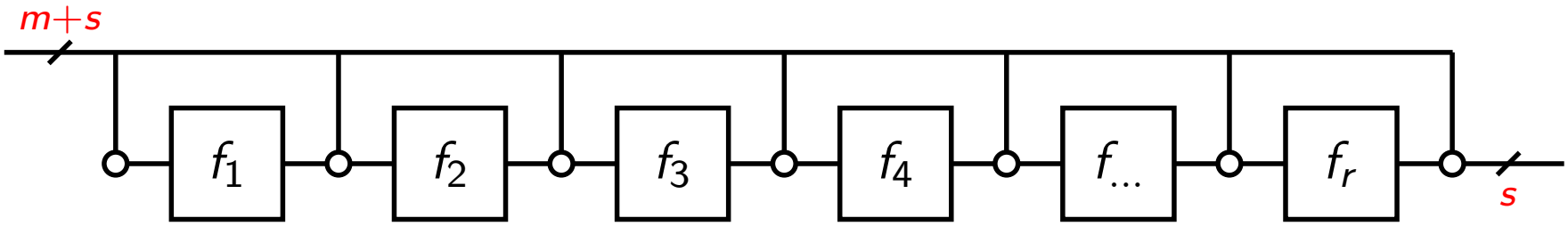
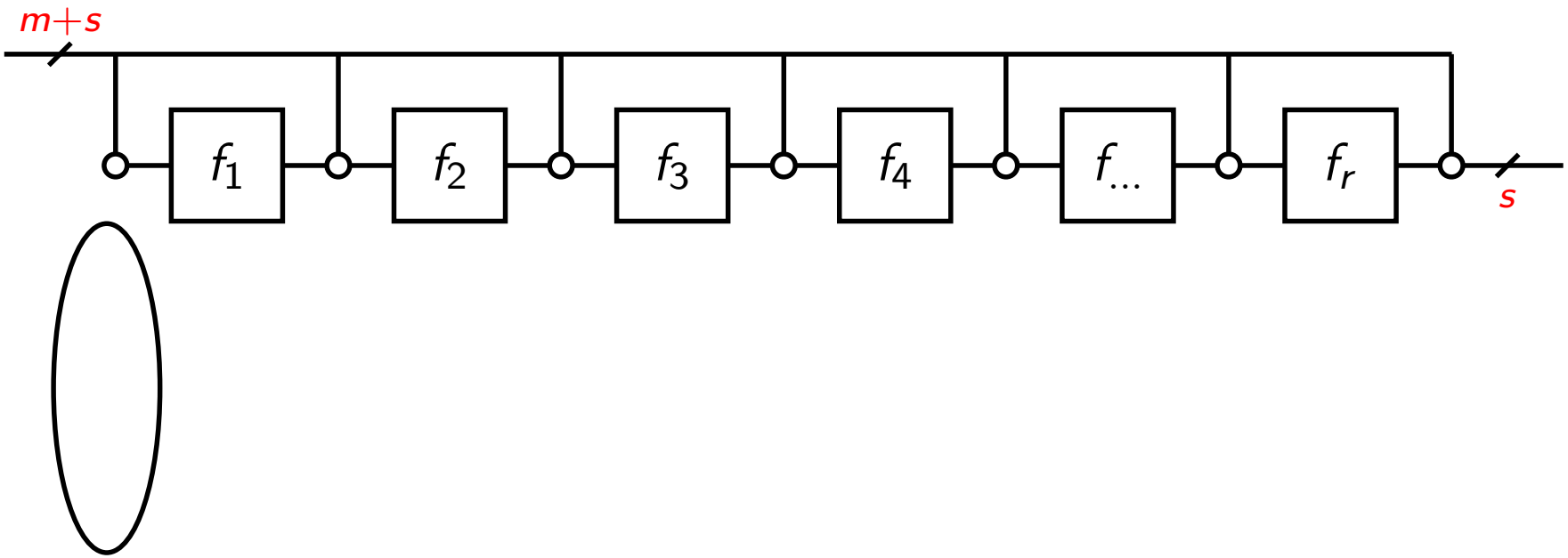
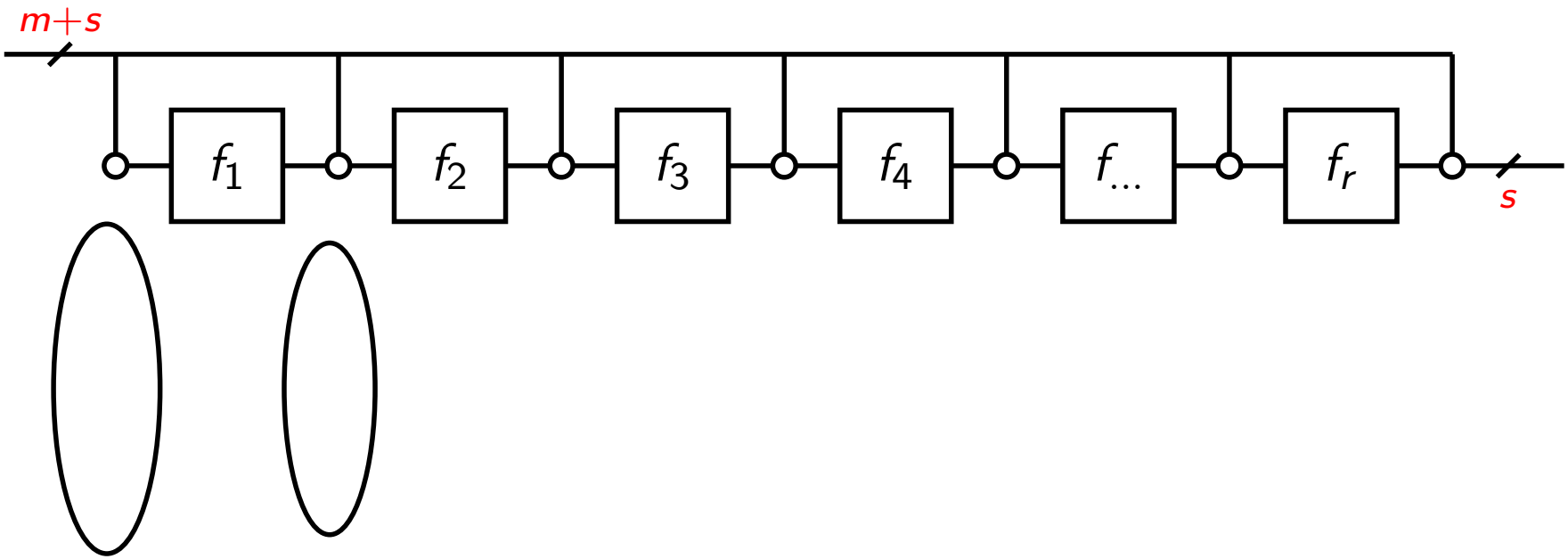
General Case



General Case



General Case



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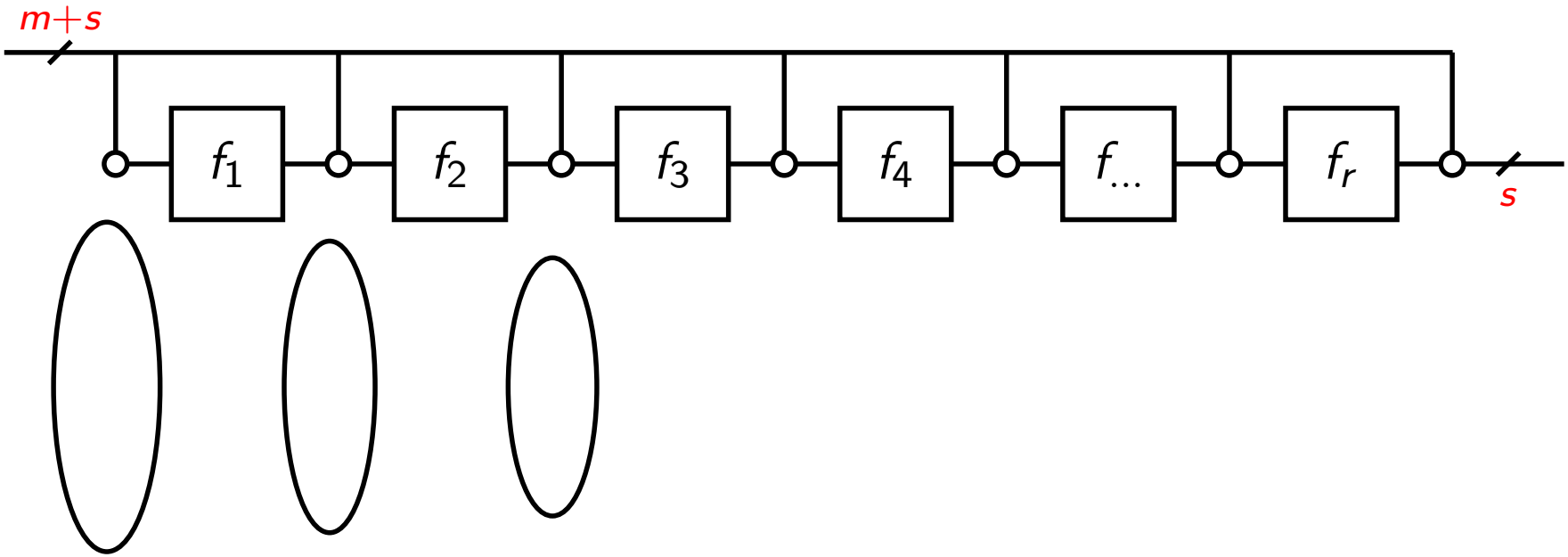

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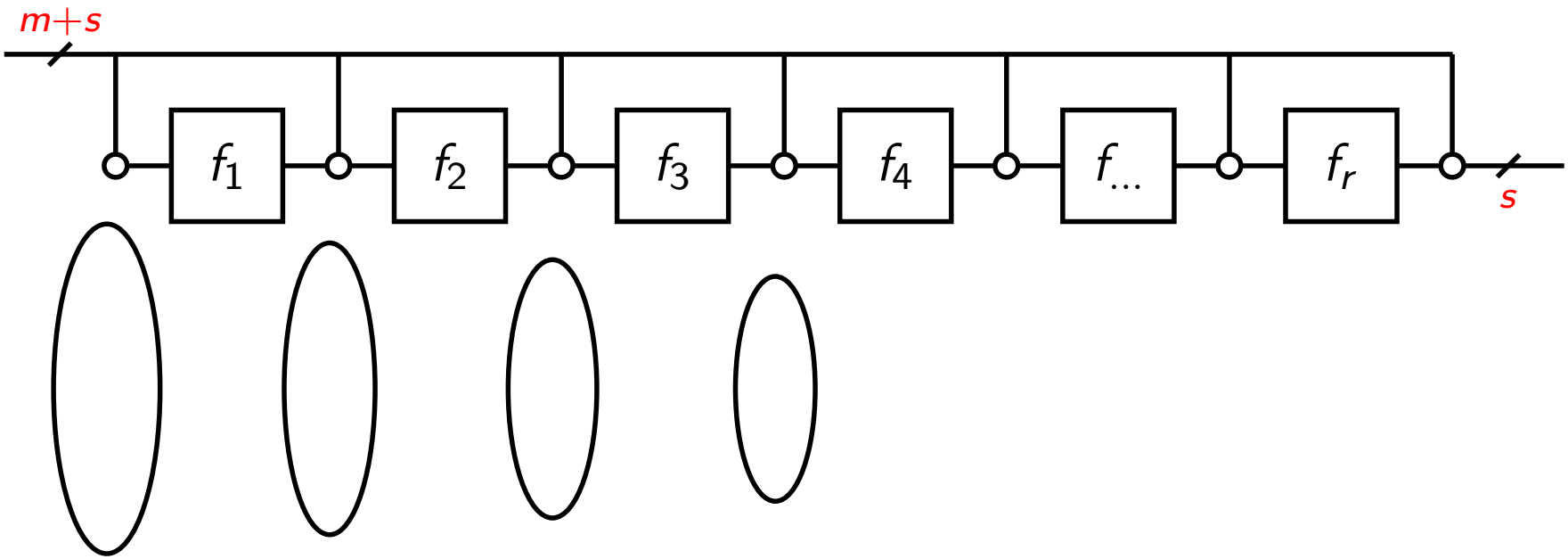

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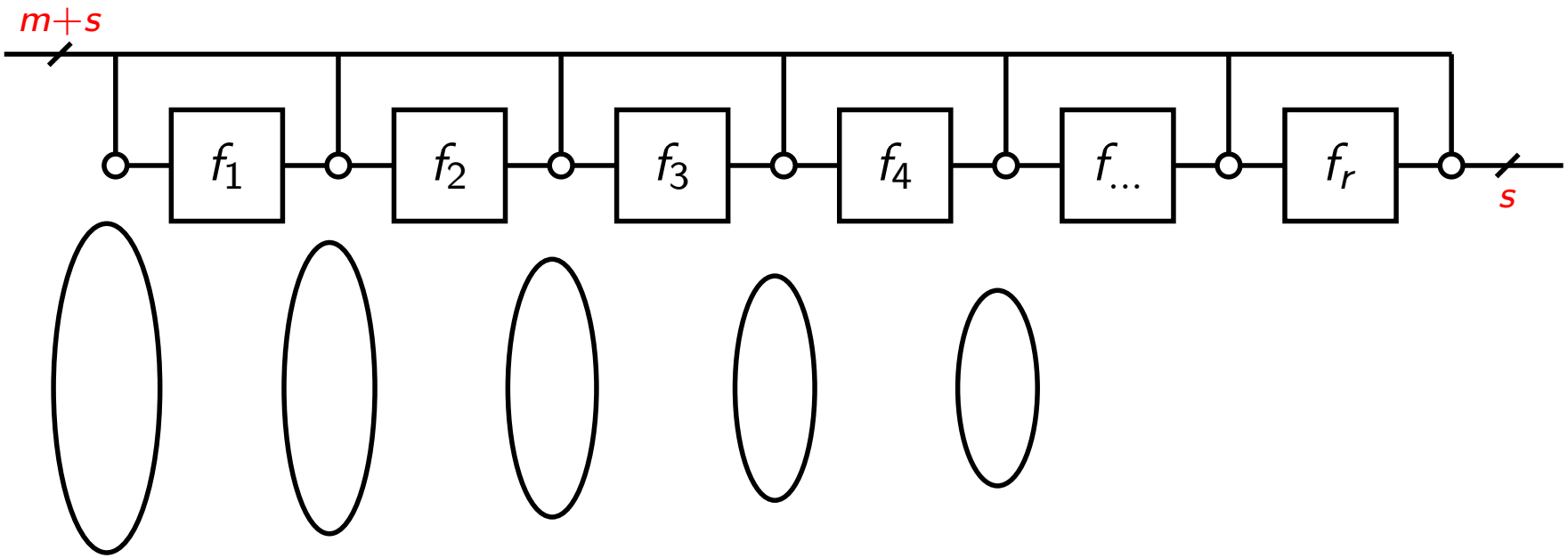

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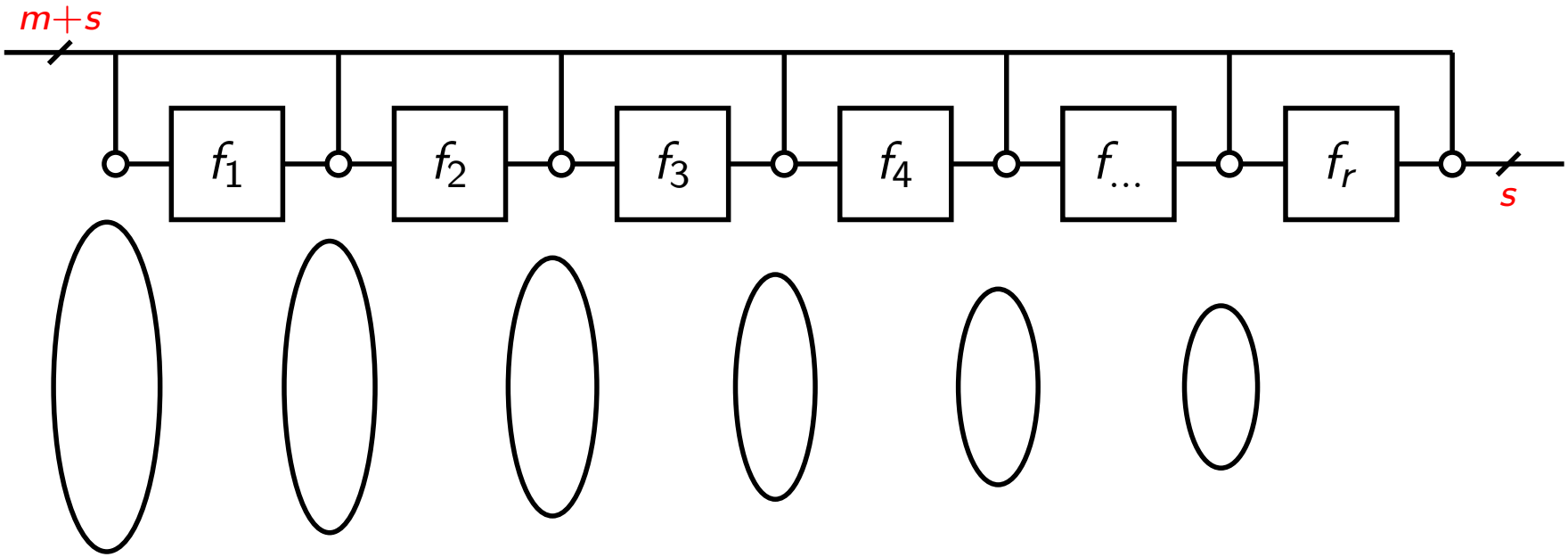

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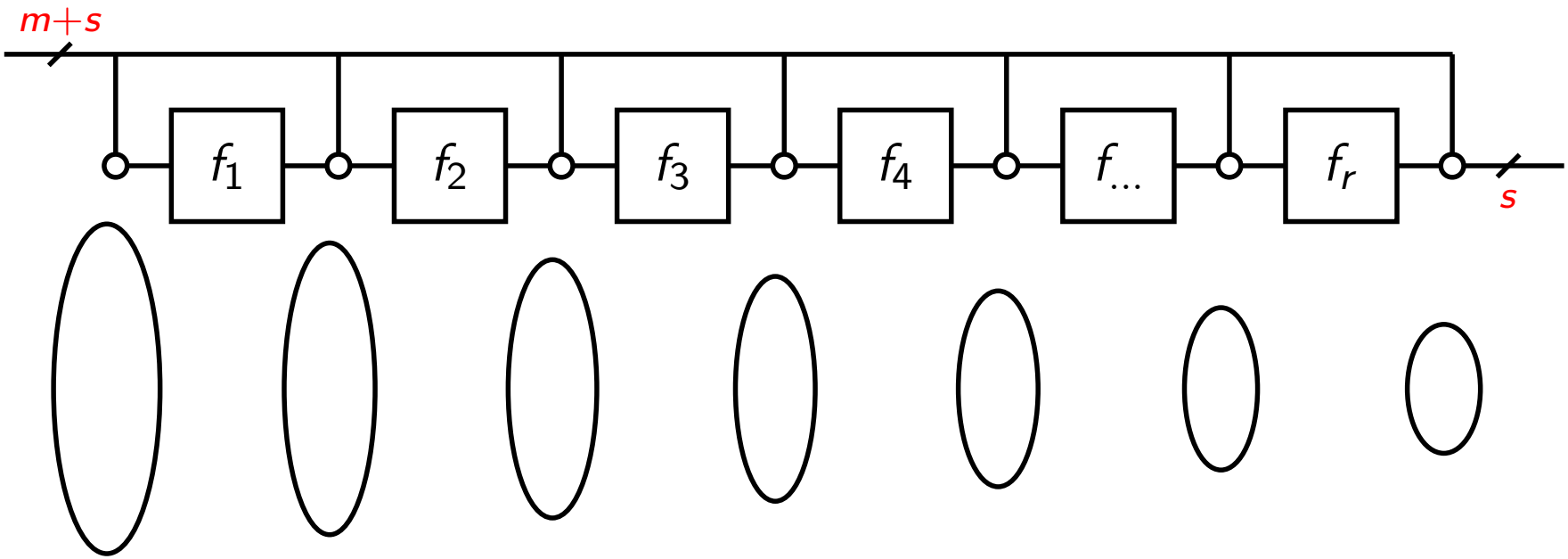

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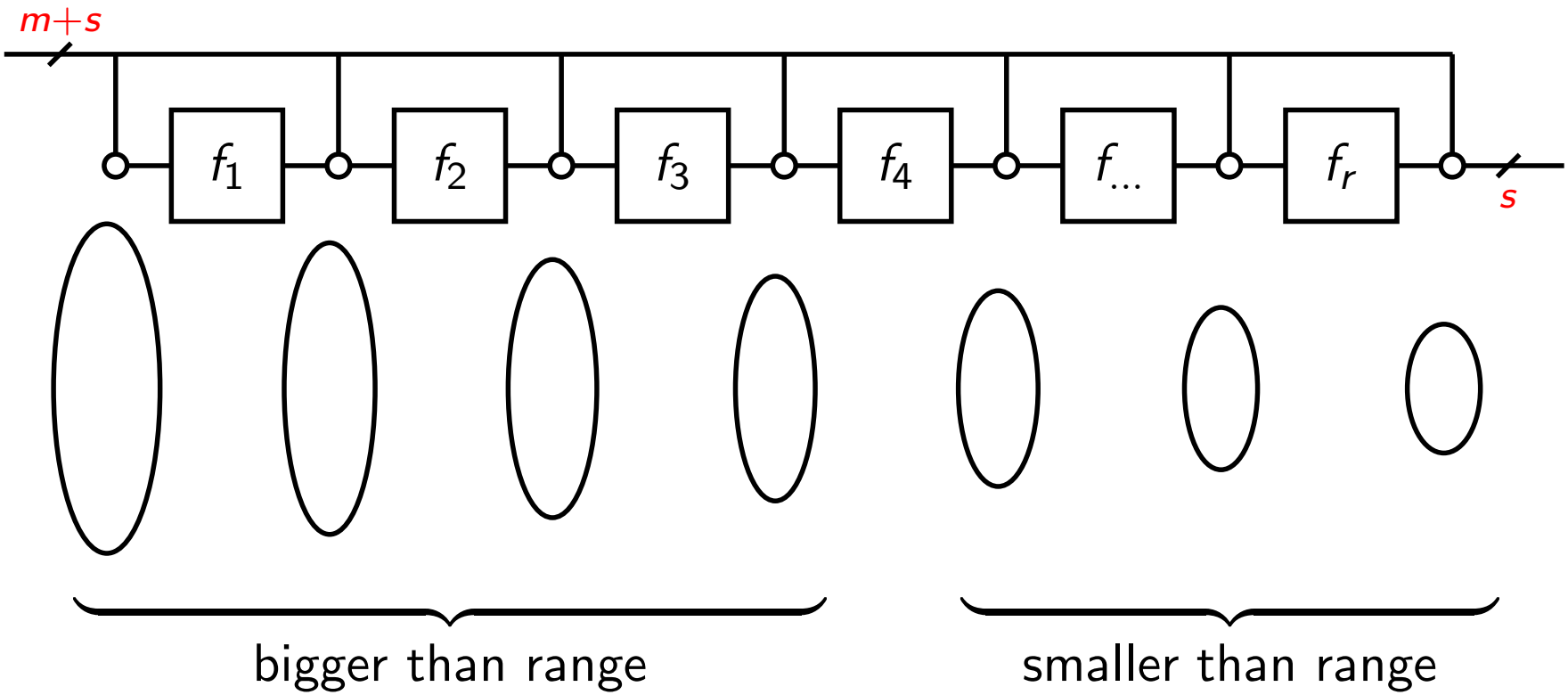

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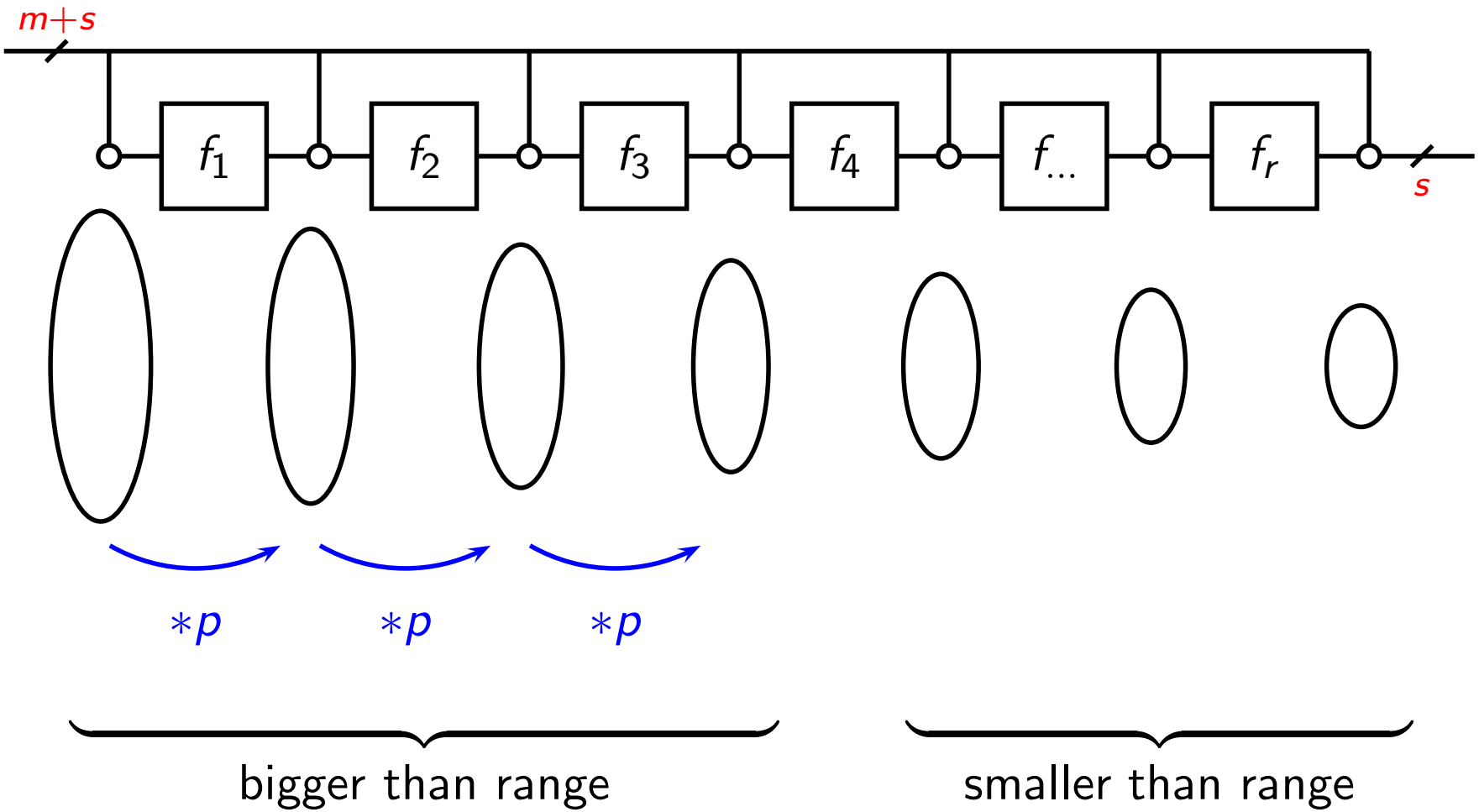

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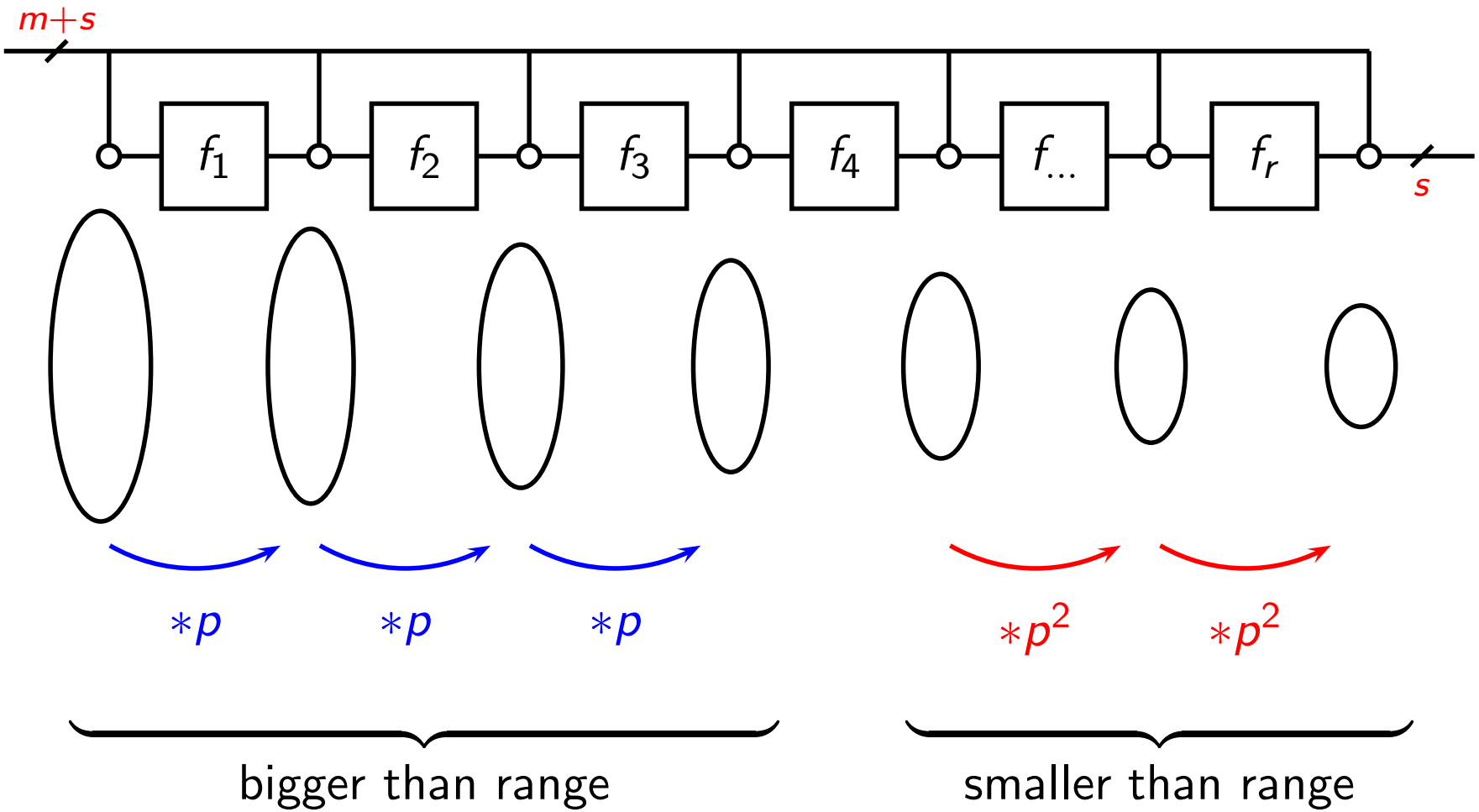

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